# 11

# Laplace Transform and Its Application

## 11.1 INTRODUCTION

Time-domain analysis is the conventional method of analysing a network. For a simple network with first-order differential equation of network variable, this method is very useful. But as the order of network variable equation increases, this method of analysis becomes very tedious. For such applications, frequency domain analysis using Laplace transform is very convenient. Time-domain analysis, also known as *classical method,* is difficult to apply to a differential equation with excitation functions which contain derivatives. Laplace transform methods prove to be superior. The Laplace transform method has the following advantages:

(1) Solution of differential equations is a systematic procedure.
(2) Initial conditions are automatically incorporated.
(3) It gives the complete solution, i.e., both complementary and particular solution in one step.

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing and probability theory.

## 11.2 LAPLACE TRANSFORMATION

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\, \mathrm{d}t$$

where $s$ is the complex frequency variable.

$$s = \sigma + j\omega$$

The function $f(t)$ must satisfy the following condition to possess a Laplace transform,

$$\int_0^{\infty} |f(t)|\, e^{-\sigma t}\, \mathrm{d}t < \infty$$

where $\sigma$ is real and positive.
The inverse Laplace transform $L^{-1}\{F(s)\}$ is

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds$$

## 11.3 LAPLACE TRANSFORMS OF SOME IMPORTANT FUNCTIONS

### 1. *Constant Function k*

The Laplace transform of a constant function is

$$L\{k\} = \int_0^\infty ke^{-st}dt = k\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{s}$$

### 2. *Function $t^n$*

The Laplace transform of $f(t)$ is

$$L\{t^n\} = \int_0^\infty t^n e^{-st}dt$$

Putting $st = x,\ dt = \dfrac{dx}{s}$

$$L\{t^n\} = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x}\frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty x^n e^{-x}dx = \frac{\sqrt{n+1}}{s^{n+1}},\ s>0,\ n+1>0$$

If $n$ is a positive integer, $\sqrt{n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

### 3. *Unit-Step Function*

The unit-step function (Fig 11.1) is defined by the equation,

$$u(t) = 1 \qquad t>0$$
$$= 0 \qquad t<0$$

The Laplace transform of unit step function is

$$L\{u(t)\} = \int_0^\infty 1\cdot e^{-st}\ dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$

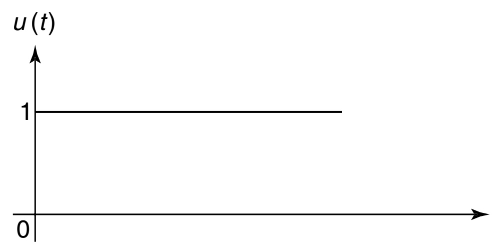



**Fig. 11.1** *Unit-step function*

### 4. *Delayed or Shifted Unit-Step Function*

The delayed or shifted unit-step function (Fig 11.2) is defined by the equation

$$u(t-a) = 1 \qquad t>a$$
$$= 0 \qquad t<a$$

The Laplace transform of $u(t-a)$ is

$$L\{u(t-a)\} = \int_a^\infty 1\cdot e^{-st}\ dt = \left[-\frac{e^{-st}}{s}\right]_a^\infty = \frac{e^{-as}}{s}$$

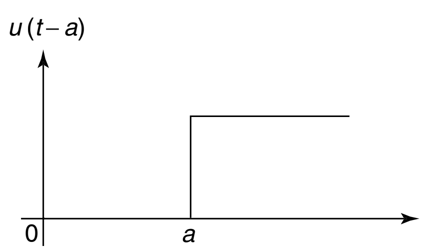



**Fig. 11.2** *Shifted unit-step function*

### 5. Unit-Ramp Function

The unit-ramp function (Fig 11.3) is defined by the equation

$$r(t) = t \qquad t > 0$$
$$= 0 \qquad t < 0$$

The Laplace transform of the unit-ramp function is

$$L\{r(t)\} = \int_0^\infty t\,e^{-st} = \frac{1}{s^2}$$

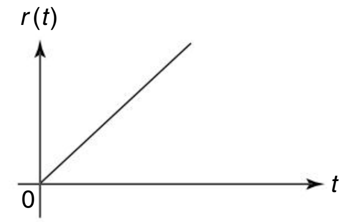



**Fig. 11.3** *Unit-ramp function*

### 6. Delayed Unit-Ramp Function

The delayed unit-ramp function (Fig 11.4) is defined by the equation

$$r\,(t-a) = t \qquad t > a$$
$$= 0 \qquad t < a$$

The Laplace transform of $r\,(t-a)$ is

$$L\{r(t-a)\} = \int_a^\infty t\,e^{-st}\,dt = \frac{e^{-as}}{s^2}$$

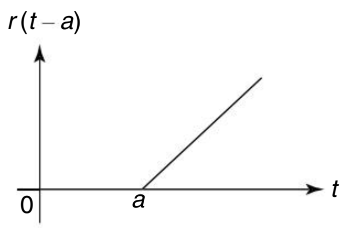



**Fig. 11.4** *Delayed unit-ramp function*

### 7. Unit-Impulse Function

The unit-impulse function (Fig 11.5) is defined by the equation

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad t = 0$$

The Laplace transform of the unit-impulse function is

$$L\{\delta(t)\} = \int_0^\infty \delta(t)\,e^{-st}\,dt = 1$$


**Fig. 11.5** *Unit-impulse function*

### 8. Exponential Function ($e^{at}$)

The Laplace transform of the exponential function (Fig 11.6) is

$$L\{e^{at}\} = \int_0^\infty e^{at}\,e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}$$

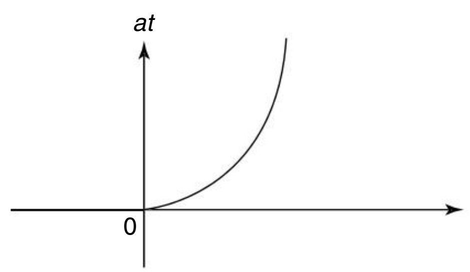



**Fig. 11.6** *Exponential function*

### 9. Sine Function

We know that $\sin\omega t = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$.

The Laplace transform of the sine function is

$$L\{\sin\omega t\} = L\left\{\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right\} = \frac{1}{2j}\left[L\{e^{j\omega t}\} - L\{e^{-j\omega t}\}\right] = \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

### 10. Cosine Function

We know that $\cos\omega t = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]$.

The Laplace transform of the cosine function is

$$L\{\cos\omega t\} = L\left\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right\} = \frac{1}{2}\left[L\{e^{j\omega t}\} + L\{e^{-j\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$$

### 11. Hyperbolic sine function

We know that $\sinh \omega t = \frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic sine function is

$$L\{\sinh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} - e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} - L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] = \frac{\omega}{s^2 - \omega^2}$$

### 12. Hyperbolic cosine function

We know that $\cosh \omega t = \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic cosine function is

$$L\{\cosh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} + e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} + L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s-\omega}\right] = \frac{s}{s^2 - \omega^2}$$

### 13. Exponentially Damped Function

Laplace transform of an exponentially damped function $e^{-at} f(t)$ is

$$L\left\{e^{-at} f(t)\,dt\right\} = \int_{0^-}^{\infty} f(t)e^{-at}\,e^{-st}\,dt = \int_{0^-}^{\infty} f(t)e^{-(s+a)t}\,dt = F(s+a)$$

Thus, the transform of the function $e^{-at} f(t)$ is obtained by putting $(s + a)$ in place of $s$ in the transform of $f(t)$.

$$L\left\{e^{-at}\sin \omega t\right\} = \frac{\omega}{(s+a)^2 + \omega^2} \qquad L\left\{e^{-at}\sinh \omega t\right\} = \frac{\omega}{(s+a)^2 - \omega^2}$$

$$L\left\{e^{-at}\cos \omega t\right\} = \frac{s+a}{(s+a)^2 + \omega^2} \qquad L\left\{e^{-at}\cosh \omega t\right\} = \frac{s+a}{(s+a)^2 - \omega^2}$$

## 11.4 PROPERTIES OF LAPLACE TRANSFORM

### 11.4.1 Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then $L\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$ where $a$ and $b$ are constants.

***Proof***

$$L\{f(t)\} = \int_0^{\infty} f(t)\,e^{-st}\,dt$$

$$L\{af_1(t) + bf_2(t)\} = \int_0^{\infty}\{af_1(t) + bf_2(t)\}e^{-st}\,dt = a\int_0^{\infty} f_1(t)e^{-st}\,dt + b\int_0^{\infty} f_2(t)e^{-st}\,dt = aF_1(s) + bF_2(s)$$

**Example 11.1** *Find the Laplace transform of* $4t^2 + \sin 3t + e^{2t}$.

**Solution**

$$L\{4t^2 + \sin 3t + e^{2t}\} = 4L\{t^2\} + L\{\sin 3t\} + L\{e^{2t}\} = 4\cdot\frac{2}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2} = \frac{8}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$$

**Example 11.2** *Find the Laplace transform of* $t^2 - e^{-2t} + \cosh^2 3t$ .

**Solution**

$$L\{t^2 - e^{-2t} + \cosh^2 3t\} = L\{t^2\} - L\{e^{-2t}\} + L\{\cosh^2 3t\} = L\{t^2\} - L\{e^{-2t}\} + \frac{1}{2}L\{1 + \cosh 6t\}$$

$$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{1}{2s} + \frac{s}{2(s^2 - 36)}$$

**Example 11.3** *Find the Laplace transform of* $(\sin 2t - \cos 2t)^2$.

**Solution** $$L\{(\sin 2t - \cos 2t)^2\} = L\{\sin^2 2t + \cos^2 2t - 2\cos 2t \sin 2t\} = L\{1 - \sin 4t\} = L\{1\} - L\{\sin 4t\}$$
$$= \frac{1}{s} - \frac{4}{s^2+16}$$

**Example 11.4** *Find the Laplace transform of* $\cos(\omega t + b)$.

**Solution** $$L\{\cos(\omega t + b)\} = L\{\cos\omega t \cos b - \sin\omega t \sin b\} = \cos b L\{\cos\omega t\} - \sin b L\{\sin\omega t\}$$
$$= \cos b \cdot \frac{s}{s^2+\omega^2} - \sin b \cdot \frac{\omega}{s^2+\omega^2}$$

### 11.4.2 Time Scaling

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

***Proof***
$$L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$
$$L\{f(at)\} = \int_0^\infty f(at) e^{-st}\, dt$$

Putting $at = x,\ dt = \frac{dx}{a}$

$$L\{(f(at)\} = \int_0^\infty f(x)\, e^{-s\left(\frac{x}{a}\right)} \frac{dx}{a} = \frac{1}{a}\int_0^\infty f(x)\, e^{-\left(\frac{s}{a}\right)x}\, dx = \frac{1}{a} F\left(\frac{s}{a}\right)$$

**Example 11.5** *If* $L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$, *find* $L\{f(2t)\}$.

**Solution**
$$L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$$

By time-scaling property,

$$L\{f(2t)\} = \frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right) = \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

**Example 11.6** *If* $L\{f(t)\} = \frac{2}{s^3}e^{-s}$, *find* $L\{f(3t)\}$.

**Solution**
$$L\{f(t)\} = \frac{2}{s^3}e^{-s}$$

By time-shifting property,

$$L\{f(3t)\} = \frac{1}{3}\frac{2}{\left(\frac{s}{3}\right)^3} e^{-\frac{s}{3}} = \frac{1}{3}\frac{54}{s^3} e^{-\frac{s}{3}} = \frac{18}{s^3} e^{-\frac{s}{3}}$$

### 11.4.3 Frequency-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{e^{-at} f(t)\} = F(s+a)$

***Proof***

$$L\{f(t)\} = \int_0^\infty f(t)t\, e^{-st}\, d$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-at} f(t) e^{-st}\, dt = \int_0^\infty f(t) e^{-(s+a)t}\, dt = F(s+a)$$

**Example 11.7** *Find the Laplace transform of* $e^{-3t}t^4$.

**Solution**

$$L\{t^4\} = \frac{4!}{s^5}$$

By frequency-shifting theorem,

$$L\{e^{-3t}t^4\} = \frac{4!}{(s+3)^5}$$

**Example 11.8** *Find the Laplace transform of* $(t+1)^2 e^t$.

**Solution**

$$L\{(t+1)^2\} = L\{t^2 + 2t + 1\} = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

By frequency-shifting theorem,

$$L\{(t+1)^2 e^t\} = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$$

**Example 11.9** *Find the Laplace transform of* $e^{4t}\sin^3 t$.

$$L\{\sin^3 t\} = \frac{1}{4} L\{3\sin t - \sin 3t\} = \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}$$

**Solution** By frequency-shifting theorem,

$$L\{e^{4t}\sin^3 t\} = \frac{3}{4[(s-4)^2+1]} - \frac{3}{4[(s-4)^2+9]} = \frac{3}{4(s^2-8s+17)} - \frac{3}{4(s^2-8s+25)} = \frac{6}{(s^2-8s+7)(s^2-8s+25)}$$

**Example 11.10** *Find the Laplace transform of cosh at cos at.*

**Solution**

$$\cosh at \cos at = \left(\frac{e^{at}+e^{-at}}{2}\right)\cos at = \frac{1}{2}(e^{at}\cos at + e^{-at}\cos at)$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\cosh at \cos at\} = \frac{1}{2} L\{e^{at}\cos at + e^{-at}\cos at\}$$

By frequency-shifting theorem,

$$L\{\cosh at \cos at\} = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right] = \frac{1}{2}\left[\frac{s-a}{s^2+2a^2-2as} + \frac{s+a}{s^2+2a^2+2as}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2+2a^2+2as)+(s+a)(s^2+2a^2-2as)}{(s^2+2a^2)^2 - 4a^2s^2}\right] = \frac{s^3}{s^4+4a^4}$$

### 11.4.4 Time-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{f(t-a)\} = e^{-as}F(s)$

***Proof***

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\, dt$$

$$L\{f(t-a)\} = \int_0^\infty f(t-a)\, e^{-st}\, dt$$

Putting $t - a = x, \quad dt = dx$

When $t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{f(t-a)\} = \int_0^\infty f(x)e^{-s(a+x)}\, dx = e^{-as}\int_0^\infty f(x)e^{-sx}\, dx = e^{-as}\int_0^\infty f(t)\, e^{-st}\, dt = e^{-as}F(s)$$

**Example 11.11** *Find the Laplace transform of* $\cos(t-a) \quad t > a$.

**Solution**

Let $f(t) = \cos t$

$$L\{f(t)\} = F(s) = \frac{s}{s^2+1}$$

By time-shifting theorem,

$$L\{\cos(t-a)\} = e^{-as}\frac{s}{s^2+1}$$

**Example 11.12** *Find the Laplace transform of* $e^{t-a} \quad t > a$.

**Solution**

Let $f(t) = e^t$

$$L\{f(t)\} = F(s) = \frac{1}{s-1}$$

By time-shifting theorem,

$$L\{e^{t-a}\} = e^{-as}\frac{1}{s-1}$$

**Example 11.13** *Find the Laplace transform of* $\sin\left(t-\frac{\pi}{4}\right) \quad t > \frac{\pi}{4}$.

**Solution**

Let $f(t) = \sin t$

$$L\{f(t)\} = F(s) = \frac{1}{s^2+1}$$

By time-shifting theorem,

$$L\left\{\sin\left(t-\frac{\pi}{4}\right)\right\} = e^{-\frac{\pi s}{4}}\frac{1}{s^2+1}$$

**Example 11.14** *Find the Laplace transform of* $(t-1)^3 \quad t > 1$.

**Solution**

Let $f(t) = t^3$

$$L\{f(t)\} = F(s) = \frac{3!}{s^4}$$

By time-shifting theorem,

$$L\left\{(t-1)^3\right\} = e^{-s}\,\frac{3!}{s^4}$$

### 11.4.5 Multiplication by *t* (Frequency-Differentiation Theorem)

If $L\{f(t)\} = F(s)$ then $L\{t\,f(t)\} = -\dfrac{d}{ds}F(s)$

***Proof***

$$L\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

Differentiating both the sides w.r.t $s$ using DUIS,

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty}\frac{\partial}{\partial s}f(t)e^{-st}\,dt$$

$$= \int_0^{\infty}(-t)\,f(t)e^{-st}\,dt = \int_0^{\infty}\{-t\,f(t)\}e^{-st}dt = -L\{t\;f(t)\}$$

$$L\{t\;f(t)\} = (-1)\frac{d}{ds}F(s)$$

**Example 11.15** *Find the Laplace transform of t sin at.*

**Solution**

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$L\{t\sin at\} = -\frac{d}{ds}L\{\sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

**Example 11.16** *Find the Laplace transform of* $t\cos^2 t$.

**Solution** $$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2}L\{1+\cos 2t\} = \frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$L\{t\cos^2 t\} = -\frac{d}{ds}L\{\cos^2 t\} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}+\frac{s}{s^2+4}\right) = -\frac{1}{2}\left[-\frac{1}{s^2}+\frac{(s^2+4)\cdot 1 - s\cdot 2s}{(s^2+4)^2}\right] = \frac{1}{2s^2}+\frac{s^2-4}{2(s^2+4)^2}$$

**Example 11.17** *Find the Laplace transform of* $t\sin^3 t$.

**Solution** $$L\{\sin^3 t\} = L\left\{\frac{3\sin t - \sin 3t}{4}\right\} = \frac{1}{4}\left(\frac{3}{s^2+1}-\frac{1}{s^2+9}\right) = \frac{3}{4}\left(\frac{1}{s^2+1}-\frac{1}{s^2+9}\right)$$

$$L\{t\sin^3 t\} = -\frac{d}{ds}L\{\sin^3 t\} = -\frac{3}{4}\frac{d}{ds}\left(\frac{1}{s^2+1}-\frac{1}{s^2+9}\right) = -\frac{3}{4}\left[\frac{-2s}{(s^2+1)^2}+\frac{2s}{(s^2+9)^2}\right] = \frac{3s}{2}\left[\frac{(s^2+9)^2-(s^2+1)^2}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{s^4+18s^2+81-s^4-2s^2-1}{(s^2+1)^2(s^2+9)^2}\right] = \frac{3s}{2}\cdot\frac{16(s^2+5)}{(s^2+1)^2(s^2+9)^2} = \frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

**Example 11.18** *Find the Laplace transform of $t \sin 2t \cosh t$.*

**Solution** $$L\{\sin 2t \cosh t\} = L\left\{\sin 2t\left(\frac{e^t + e^{-t}}{2}\right)\right\} = \frac{1}{2}L\{e^t \sin 2t + e^{-t}\sin 2t\}$$

$$= \frac{1}{2}\left[\frac{2}{(s-1)^2+4} + \frac{2}{(s+1)^2+4}\right] = \frac{1}{s^2-2s+5} + \frac{1}{s^2+2s+5}$$

$$L\{t \sin 2t \cosh t\} = -\frac{d}{ds}L\{\sin 2t \cosh t\} = -\frac{d}{ds}\left(\frac{1}{s^2-2s+5} + \frac{1}{s^2+2s+5}\right) = \frac{2s-2}{(s^2-2s+5)^2} + \frac{2s+2}{(s^2+2s+5)^2}$$

### 11.4.6 Division by t (Frequency-Integration Theorem)

If $L\{f(t)\} = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\, ds$

***Proof***

$$L\{f(t)\} = F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

Integrating both the sides w.r.t $s$ from $s$ to $\infty$,

$$\int_s^{\infty} F(s)\, ds = \int_s^{\infty}\int_0^{\infty} f(t) e^{-st} dt\, ds$$

Since $s$ and $t$ are independent variables, interchanging the order of integration,

$$\int_s^{\infty} F(s) ds = \int_0^{\infty}\left[\int_s^{\infty} f(t) e^{-st} ds\right] dt = \int_0^{\infty}\left[\frac{1}{-t} f(t) e^{-st}\right]_s^{\infty} dt = \int_0^{\infty} \frac{f(t)}{t} e^{-st} dt$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s) ds$$

**Example 11.19** *Find the Laplace transform of $\frac{1-e^{-t}}{t}$.*

**Solution** $$L\{1-e^{-t}\} = \frac{1}{s} - \frac{1}{s+1}$$

$$L\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^{\infty} L\{1-e^{-t}\} ds = \int_s^{\infty}\left(\frac{1}{s} - \frac{1}{s+1}\right) ds = \left|\log s - \log(s+1)\right|_s^{\infty}$$

$$= \left[\log \frac{s}{s+1}\right]_s^{\infty} = \log\left[\frac{1}{1+\frac{1}{s}}\right]_s^{\infty} = \log 1 - \log\left(\frac{1}{1+\frac{1}{s}}\right) = -\log\frac{s}{s+1} = \log\frac{s+1}{s}$$

**Example 11.20** *Find the Laplace transform of* $\dfrac{e^{-at}-e^{-bt}}{t}$.

**Solution**

$$L\{e^{-at}-e^{-bt}\}=\frac{1}{s+a}-\frac{1}{s+b}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\int_s^\infty L\{e^{-at}-e^{-bt}\}\,ds=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds=\left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$=\left[\log\frac{s+a}{s+b}\right]_s^\infty=\left[\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right]_s^\infty=\log 1-\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}=-\log\frac{s+a}{s+b}=\log\frac{s+b}{s+a}$$

**Example 11.21** *Find the Laplace transform of* $\dfrac{\sinh t}{t}$.

**Solution**

$$L\{\sinh t\}=L\left\{\frac{e^{t}-e^{-t}}{2}\right\}=\frac{1}{2}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)$$

$$L\left\{\frac{\sinh t}{t}\right\}=\int_s^\infty L\{\sinh t\}\,ds=\frac{1}{2}\int_s^\infty\left(\frac{1}{s-1}-\frac{1}{s+1}\right)ds=\frac{1}{2}\left[\log(s-1)-\log(s+1)\right]_s^\infty=\frac{1}{2}\left[\log\frac{s-1}{s+1}\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right]_s^\infty=\frac{1}{2}\left(\log 1-\log\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right)=-\frac{1}{2}\log\frac{s-1}{s+1}=\frac{1}{2}\log\frac{s+1}{s-1}$$

**Example 11.22** *Find the Laplace transform of* $\dfrac{\cos h2t\ \sin 2t}{t}$.

**Solution**

$$L\left\{\frac{\cosh 2t\sin 2t}{t}\right\}=L\left\{\left(\frac{e^{2t}+e^{-2t}}{2t}\right)\sin 2t\right\}=\frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\}+L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right]$$

$$L\{\sin 2t\}=\frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\}=\int_s^\infty L\{\sin 2t\}ds=\int_s^\infty\frac{2}{s^2+4}ds=\left[\tan^{-1}\frac{s}{2}\right]_s^\infty=\frac{\pi}{2}-\tan^{-1}\frac{s}{2}=\cot^{-1}\frac{s}{2}$$

By first shifting theorem,

$$L\left\{\frac{\cosh 2t\sin 2t}{t}\right\}=\frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\}+L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right]=\frac{1}{2}\left[\cot^{-1}\left(\frac{s-2}{2}\right)+\cot^{-1}\left(\frac{s+2}{2}\right)\right]$$

### 11.4.7 Time-Differentiation Theorem: Laplace Transform of Derivatives

If $L\{f(t)\}=F(s)$ then

$$L\{f'(t)\}=sF(s)-f(0)$$

$$L\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$$

In general,

$$L\{f''(t)\} = s''F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0)\ldots - f^{(n-1)}(0)$$

***Proof***

$$L\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt$$

Integrating by parts,

$$L\{f'(t)\} = \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty (-s)f(t)e^{-st}dt = -f(0) + s\int_0^\infty f(t)e^{-st}dt = -f(0) + sL\{f(t)\}$$

Similarly,

$$L\{f''\ t\} = -f' + sL\{f'\ t\} = -f'(0) + s[-f(0) + sL\{f(t)\}] = -f'(0) - sf(0) + s^2L\{f(t)\}$$

In general, $L\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0)\ldots - f^{n-1}(0)$

**Example 11.23** *Find $L\{f(t)\}$ and $L\{f'(t)\}$ of $f(t) = \dfrac{\sin t}{t}$.*

**Solution**

$$L\{f(t)\} = F(s) = L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}ds = \int_s^\infty \frac{1}{s^2+1}ds = \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$L\{f'(t)\} = sF(s) - f(0) = s\cot^{-1} s - \lim_{t\to 0}\frac{\sin t}{t} = s\cot^{-1} s^{-1}$$

**Example 11.24** *Find $L\{f(t)\}$ and $L\{f'(t)\}$ of the following function:*

$$f(t) = 3 \qquad 0 \le t < 5$$
$$= 0 \qquad t > 5.$$

**Solution**

$$L\{f(t)\} = F(s) = \int_s^\infty e^{-st}f(t)dt = \int_0^5 e^{-st}\cdot 3dt + \int_5^\infty 0\cdot dt = 3\left[\frac{e^{-st}}{-s}\right]_0^5 + 0 = \frac{-3}{s}(e^{-5s}-1) = \frac{3}{s}(1-e^{-5s})$$

$$L\{f'(t)\} = sF(s) - f(0) = s\cdot\frac{3}{s}(1-e^{-5s}) - 3 = -3e^{-5s}$$

**Example 11.25** *Find $L\{f(t)\}$ and $L\{f'(t)\}$ of $f(t) = e^{-5t}\sin t$.*

**Solution**

$$L\{f(t)\} = F(s) = L\{e^{-5t}\sin t\} = \frac{1}{(s+5)^2+1}$$

$$L\{f'(t)\} = sF(s) - f(0) = s\cdot\frac{1}{s^2+10s+26} - e^0\sin 0 = \frac{s}{s^2+10s+26}$$

**Example 11.26** *Find $L\{f(t)\}$ and $L\{f'(t)\}$ of the following function:*

$$f(t) = t \qquad 0 \le t < 3$$
$$= 6 \qquad t > 3.$$

**Solution** $$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \int_0^3 e^{-st}t\,dt + \int_3^\infty e^{-st}\cdot 6\,dt = \left[\frac{e^{-st}}{-s}\cdot t\right]_0^3 - \left[\frac{e^{-st}}{s^2}\right]_0^3 + 6\left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6}{s}e^{-3s} = \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$$

$$L'\{f(t)\} = sF(s) = f(0) = \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right)$$

### 11.4.8 Time-Integration Theorem: Laplace Transform of Integral

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$

***Proof*** $$L\left\{\int_0^t f(t)dt\right\} = \int_0^\infty \left\{\int_0^t f(t)\,dt\right\} e^{-st}dt$$

Integrating by parts,

$$L\left\{\int_0^t f(t)dt\right\} = \left[\int_0^t f(t)dt\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty\left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{d}{dt}\int_0^t f(t)dt\right)\right]dt = \int_0^\infty \frac{1}{s}f(t)e^{-st}dt = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$$

**Example 11.27** *Find the Laplace transform of* $\int_0^t e^{-2t}t^3 dt$.

**Solution**

$$L\{e^{-2t}t^3\} = \frac{3!}{(s+2)^4} = \frac{6}{(s+2)^4}$$

$$L\left\{\int_0^t e^{-2t}t^3 dt\right\} = \frac{1}{s}L\left\{e^{-2t}t^3\right\} = \frac{6}{s(s+2)^4}$$

**Example 11.28** *Find the Laplace transform of* $\int_0^t t\cosh t\,dt$.

**Solution** $$L\{t\cosh t\} = L\left\{t\left(\frac{e^t + e^{-t}}{2}\right)\right\} = \frac{1}{2}L\{te^t + te^{-t}\} = \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right] = \frac{1}{2}\cdot\frac{2(s^2+1)}{(s^2-1)^2} = \frac{s^2+1}{(s^2-1)^2}$$

$$L\left\{\int_0^t t\cosh t\,dt\right\} = \frac{1}{s}L\{t\cosh t\} = \frac{s^2+1}{s(s^2-1)^2}$$

**Example 11.29** *Find the Laplace transform of the* $\int_0^t te^{-4t}\sin 3t\,dt$.

**Solution**

$$L\{t\sin 3t\} = -\frac{d}{ds}L\{\sin 3t\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\{te^{-4t}\sin 3t\} = \frac{6(s+4)}{[(s+4)^2+9]^2} = \frac{6(s+4)}{(s^2+8s+25)^2}$$

$$L\left\{\int_0^t t\,e^{-4t}\sin 3t\;dt\right\} = \frac{1}{s}L\{t\,e^{-4t}\sin 3t\} = \frac{6(s+4)}{s(s^2+8s+25)^2}$$

**Example 11.30** *Find the Laplace transform of* $e^{-4t}\int_0^t t\sin 3t\,dt$.

**Solution**

$$L\{t\sin 3t\} = -\frac{d}{ds}L\{\sin 3t\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\left\{\int_0^t t\sin 3t\right\} = \frac{1}{s}L\{t\sin 3t\} = \frac{6}{(s^2+9)^2}$$

$$L\left\{e^{-4t}\int_0^t t\sin 3t\right\} = \frac{6}{[(s+4)^2+9]^2} = \frac{6}{(s^2+8s+25)^2}$$

### 11.4.9 Initial Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

***Proof*** We know that,

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\,e^{-st}dt + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt + f(0) = \int_0^\infty \lim_{s\to\infty}[f'(t)e^{-st}]dt + f(0) = 0 + f(0) = f(0) = \lim_{t\to 0} f(t)$$

### 11.4.10 Final Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim_{t\to\infty} f(t) = \lim_{s\to\infty} sF(s)$

***Proof*** We know that

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\,e^{-st}dt + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to 0}\int_0^\infty f'(t)\,e^{-st}dt + f(0) = \int_0^\infty \lim_{s\to 0}\left[f'(t)e^{-st}\right]dt + f(0) = \int_0^\infty f'(t)dt + f(0)$$

$$= \left|f(t)\right|_0^\infty + f(0) = \lim_{t\to\infty} f(t) - f(0) + f(0) = \lim_{t\to\infty} f(t)$$

**Example 11.31** *Verify the initial and final value theorems for* $e^{-t}(t+1)^2$.

**Solution**

$$f(t) = e^{-t}(t+1)^2 = e^{-t}(t^2+2t+1)$$

$$F(s) = \frac{2}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{1}{s+1}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{2s}{(s+1)^2} + \frac{s}{s+1}$$

$$\lim_{t\to 0} f(t) = 1$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{\frac{2}{s^2}}{\left(1+\frac{1}{s}\right)^3} + \frac{\frac{2}{s}}{\left(1+\frac{1}{s}\right)^2} + \frac{1}{1+\frac{1}{s}}\right] = 1$$

Hence, the initial value theorem is verified.

$$\lim_{t\to\infty} f(t) = 0$$

$$\lim_{s\to 0} sF(s) = 0$$

Hence, the final value theorem is verified.

**Example 11.32** *Verify the initial and final value theorems for* $e^{-t}(t^2+\cos 3t)$.

**Solution**

$$f(t) = e^{-t}(t^2+\cos 3t)$$

$$F(s) = \frac{2}{(s+1)^3} + \frac{s+1}{(s+1)^2+9}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{s(s+1)}{(s+1)^2+9}$$

$$\lim_{t\to 0} f(t) = 1$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{\frac{2}{s^2}}{\left(1+\frac{1}{s}\right)^3} + \frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{1}{s}\right)^2+\frac{9}{s^2}}\right] = 1$$

Hence, the initial value theorem is verified.

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(t^2+\cos 3t)e^{-t} = 0$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{2s}{(s+1)^3} + \frac{s(s+1)}{(s+1)^2+9}\right] = 0$$

Hence, the final value theorem is verified.

**Example 11.33** *Find the initial and final values of the function whose Laplace transform is*

$$F(s) = \frac{2s+1}{s^3+6s^2+11s+6}.$$

**Solution**

$$F(s) = \frac{2s+1}{s^3+6s^2+11s+6}$$

$$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{2s^2+s}{s^3+6s^2+11s+6} = \lim_{s\to\infty} \frac{\frac{2}{s}+\frac{1}{s^2}}{1+\frac{6}{s}+\frac{11}{s^2}+\frac{6}{s^3}} = 0$$

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{2s^2+s}{s^3+6s^2+11s+6} = 0$$

**Example 11.34** *Find the final value of the function whose Laplace transform is* $I(s) = \dfrac{s+6}{s(s+3)}$.

**Solution**

$$I(s) = \frac{s+6}{s(s+3)}$$

$$I(\infty) = \lim_{s\to 0} s\,I(s) = \lim_{s\to 0} \frac{s+6}{s+3} = 2$$

## 11.5 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

A function $f(t)$ is said to be periodic if there exists a constant $T(T > 0)$ such that $f(t+T) = f(t)$, for all values of $t$.

$$f(t+2T) = f(t+T+T) = f(t+T) = f(t)$$

In general, $f(t+nT) = f(t)$ for all $t$, where $n$ is an integer (positive or negative) and $T$ is the period of the function.

If $f(t)$ is a piecewise continuous periodic function with period $T$ then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T f(t)e^{-st}\,dt$$

***Proof***

$$L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\,dt = \int_0^T f(t)e^{-st}\,dt + \int_T^\infty f(t)e^{-st}\,dt$$

In the second integral, putting $t = x+T,\ dt = dx$

When

$$t = T, \quad x = 0$$

$$t \to \infty, \quad x \to \infty$$

$$L\{f(t)\} = \int_0^T f(t)e^{-st}\,dt + \int_0^\infty f(x+T)e^{-s(x+T)}\,dx$$

$$= \int_0^T f(t)e^{-st}\,dt + e^{-Ts}\int_0^\infty f(x)e^{-sx}\,dx$$

$$= \int_0^T f(t)e^{-st}\,dt + e^{-Ts}\int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^T f(t)e^{-st}\,dt + e^{-Ts}L\{f(t)\}$$

$$(1-e^{-Ts})L\{f(t)\} = \int_0^T f(t)e^{-st}\,dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T f(t)e^{-st}\,dt$$

**Example 11.35** *Find the Laplace transform of the waveform shown in Fig. 11.7.*

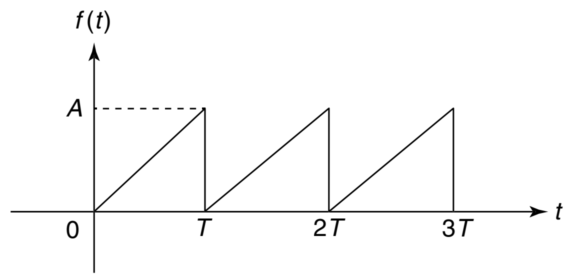



**Fig. 11.7**

**Solution** The function $f(t)$ is a periodic function with period $T$.

$$f(t) = \frac{At}{T} \qquad 0<t<t$$

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T f(t)e^{-st}\,dt$$

$$= \frac{1}{1-e^{-Ts}}\int_0^T \frac{At}{T}e^{-st}\,dt$$

$$= \frac{1}{1-e^{-Ts}}\frac{A}{T}\int_0^T te^{-st}\,dt$$

$$= \frac{A}{T(1-e^{-Ts})}\left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^T$$

$$= \frac{A}{T(1-e^{-Ts})}\left(-T\frac{e^{-Ts}}{s} - \frac{e^{-Ts}}{s^2} + \frac{1}{s^2}\right)$$

$$= \frac{A}{T(1-e^{-Ts})}\left[-\frac{Te^{-Ts}}{s} + \frac{1}{s^2}(1-e^{-Ts})\right]$$

$$= \frac{A}{Ts^2} - \frac{Ae^{-Ts}}{s(1-e^{-Ts})}$$

**Example 11.36** *Find the Laplace transform of the waveform shown in Fig. 11.8.*

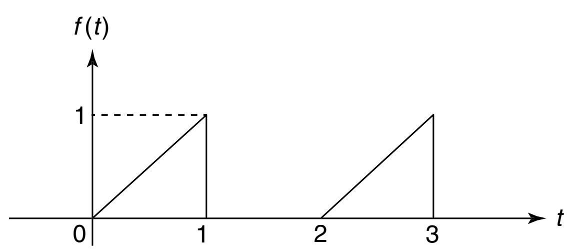



**Fig. 11.8**

**Solution** The function $f(t)$ is a periodic function with period 2.

$$f(t) = t \quad 0 < t < 1$$
$$= 0 \quad 1 < t < 2$$

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 f(t)e^{-st}\,dt$$
$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 te^{-st}\,dt + \int_1^2 0\cdot e^{-st}dt\right]$$
$$= \frac{1}{1-e^{-2s}}\left[\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\right]_0^1$$
$$= \frac{1}{1-e^{-2s}}\left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right)$$
$$= \frac{1}{s^2(1-e^{-2s})}(1-e^{-s}-se^{-s})$$

**Example 11.37** *Find the Laplace transform of the waveform shown in Fig. 11.9.*

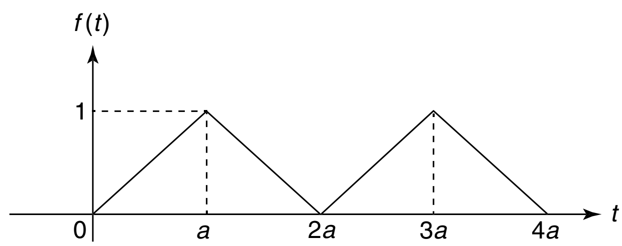



**Fig. 11.9**

**Solution** The function $f(t)$ is a periodic function with period $2a$.

$$f(t) = \frac{t}{a} \qquad 0 < t < a$$
$$= \frac{1}{a}(2a-t) \qquad a < t < 2a$$

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} f(t)e^{-st}\,dt$$
$$= \frac{1}{1-e^{-2as}}\left[\int_a^a \frac{t}{a}e^{-st}dt + \int_a^{2a}\frac{1}{a}(2a-t)e^{-st}\,dt\right]$$
$$= \frac{1}{a(1-e^{-2as})}\left\{\left[\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\right]_0^a + \left[\frac{e^{-st}}{-s}(2a-t) + \frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$
$$= \frac{1}{a(1-e^{-2as})}\left(-\frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2}\right)$$

$$= \frac{-2e^{-as} + 1 + e^{-2as}}{as^2(1 - e^{-2as})}$$

$$= \frac{(1 - e^{-as})^2}{as^2(1 - e^{-as})(1 + e^{-as})}$$

$$= \frac{1 - e^{-as}}{as^2(1 + e^{-as})}$$

$$= \frac{e^{\frac{as}{2}} - e^{\frac{-as}{2}}}{as^2\left(e^{\frac{as}{2}} + e^{\frac{-as}{2}}\right)}$$

$$= \frac{\tanh\left(\frac{as}{2}\right)}{as^2}$$

**Example 11.38** *Find the Laplace transform of the waveform shown in Fig. 11.10.*

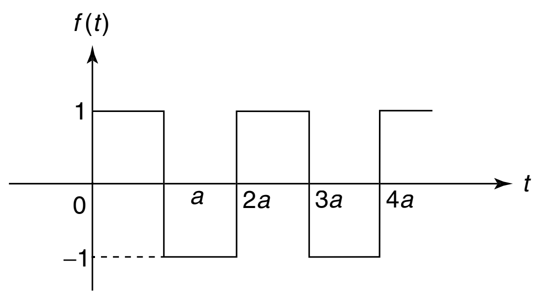



**Fig. 11.10**

**Solution** The function $f(t)$ is periodic with period with period $2a$.

$$f(t) = 1 \qquad 0 < t < a$$

$$= -1 \qquad a < t < 2a$$

$$L\{f(t)\} = \frac{1}{1 - e^{-2as}} \int_0^{2a} f(t) e^{-st}\, dt$$

$$= \frac{1}{1 - e^{-2as}} \left[\int_0^a e^{-st}\, dt + \int_a^{2a} e^{-st}(-1)\, dt\right]$$

$$= \frac{1}{1 - e^{-2as}} \left\{\left[\frac{e^{-st}}{-s}\right]_0^a + \left[\frac{e^{-st}}{s}\right]_a^{2a}\right\}$$

$$= \frac{1}{1 - e^{-2as}} \left(-\frac{e^{-as}}{-s} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right)$$

$$= \frac{(1 - e^{-as})^2}{s(1 + e^{-as})(1 - e^{-as})}$$

$$= \frac{1-e^{-as}}{s(1+e^{-as})}$$

$$= \frac{1}{s} \cdot \frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{\left(e^{\frac{as}{2}} + e^{-\frac{as}{2}}\right)}$$

$$= \frac{1}{s} \tanh\left(\frac{as}{2}\right)$$

**Example 11.39** *Find the Laplace transform of the waveform shown in Fig. 11.11.*

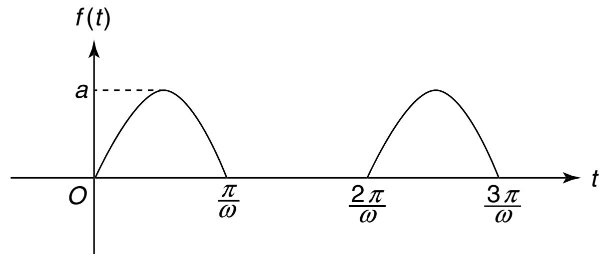



**Fig. 11.11**

**Solution** The function $f(t)$ is known as a half-sine wave rectifier function with period $\frac{2\pi}{\omega}$.

$$f(t) = a \sin \omega t \quad 0 < t < \frac{\pi}{\omega}$$
$$= 0 \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$$

The function $f(t)$ is a periodic function.

$$L\{f(t)\} = \frac{1}{1-e^{-\left(\frac{2\pi}{\omega}\right)s}} \int_0^{\frac{2\pi}{\omega}} f(t) e^{-st}\, dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left( \int_0^{\frac{\pi}{\omega}} a \sin \omega t\, e^{-st} dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} 0 \cdot e^{-st}\, dt \right)$$

$$= \frac{a}{1-e^{-\frac{2\pi s}{\omega}}} \left[ \frac{1}{s^2+\omega^2} \cdot e^{-st}(-s \sin \omega t - \omega \cos \omega t) \right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{a}{1-e^{-\frac{2\pi s}{\omega}}} \cdot \frac{1}{s^2+\omega^2} \left[ e^{-\frac{\pi s}{\omega}}(\omega) + \omega \right]$$

$$= \frac{a\omega\left(1+e^{\frac{-\pi s}{\omega}}\right)}{\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(1-e^{\frac{-\pi s}{\omega}}\right)} \cdot \frac{1}{s^2+\omega^2}$$

$$= \frac{a\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)} \cdot \frac{1}{s^2+\omega^2}$$

**Example 11.40** *Find the Laplace transform of*

$$f(t) = t^2 \qquad 0 < t < 2$$

if

$$f(t) = f(t+2).$$

**Solution** The function *f*(*t*) is a periodic function with period 2.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,dt = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}\cdot t^2\,dt$$

$$= \frac{1}{1-e^{-2s}}\left[t^2\cdot\left(\frac{e^{-st}}{-s}\right) - 2t\left(\frac{e^{-st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right)\right]_0^2$$

$$= \frac{1}{1-e^{-2s}}\left(-4\frac{e^{-2s}}{s} - 4\frac{e^{-2s}}{s^2} - 2\frac{e^{-2s}}{s^3} + \frac{2}{s^3}\right)$$

$$= \frac{1}{\left(1-e^{-2s}\right)s^3}\left(2 - 2e^{-2s} - 4se^{-2s} - 4s^2e^{-2s}\right)$$

**Example 11.41** *Find the Laplace transform of*

$$f(t) = e^t \qquad 0 < t < 2\pi$$

if

$$f(t) = f(t+2\pi).$$

**Solution** The function *f*(*t*) is a periodic function with period $2\pi$.

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}e^{t}\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{(1-s)t}\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{(1-s)t}}{1-s}\right]_0^{2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{(1-s)2\pi}}{1-s} - \frac{1}{1-s}\right]$$

$$= \frac{e^{(1-s)2\pi} - 1}{(1-e^{-2\pi s})(1-s)}$$

**Example 11.42** *Find the Laplace transform of the function shown in Fig. 11.12.*

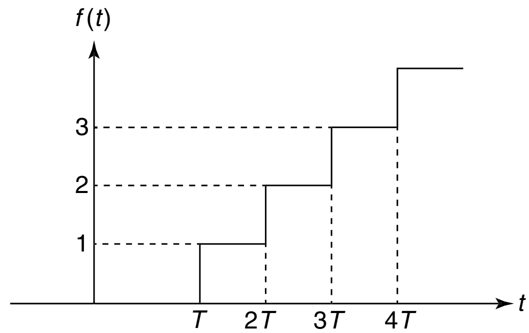



**Fig. 11.12**

**Solution** The function $f(t)$ can be represented in terms of Heaviside unit step function.

$$f(t) = [u(t-T) - u(t-2T)] + 2[u(t-2T) - u(t-3T) + 3[u(t-3T) - u(t-4t)] + \ldots\infty$$

$$= u(t-T) + u(t-2T) + u(t-3T) + \ldots\infty$$

$$L\{f(t)\} = L\{u(t-T) + u(t-2T) + u(t-3T) + \ldots\}$$

$$= \frac{1}{s}e^{-Ts} + \frac{1}{s}e^{-2Ts} + \frac{1}{s}e^{-3Ts} + \ldots$$

$$= \frac{1}{s}\left[e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \ldots\right]$$

$$= \frac{e^{-Ts}}{s(1-e^{-Ts})}$$

## 11.6 WAVEFORM SYNTHESIS

Any waveform can be constructed with unit step, unit ramp and unit impulse functions, etc. We know the Laplace transforms of these special functions. Hence, we can find the Laplace transform of any function in terms of Laplace transform of these functions.

There is another way of synthesising the waveforms. Any function can be expressed in terms of a gate function. The gate function is shown in Fig. 11.13.

This function can be expressed in terms of unit-step functions.

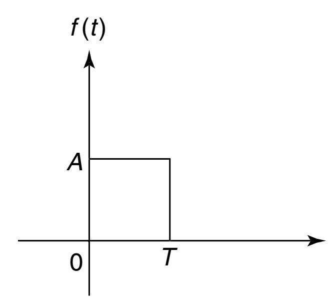



**Fig. 11.13** *Gate function*

$$f(t) = Au(t) - Au(t-T)$$

**Example 11.43** *Find the Laplace transform of the unit-doublet function.*

**Solution** The unit-doublet function $\delta'(t)$ is shown in Fig. 11.14.

$$\delta'(t) = \frac{d}{dt}\delta(t)$$

$$L\{\delta'(t)\} = L\left\{\frac{d}{dt}\delta(t)\right\} = sL\{\delta(t)\} = s(1) = s$$

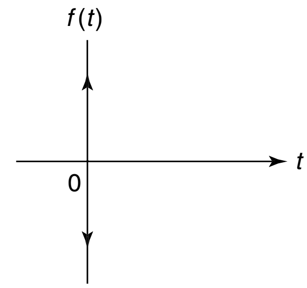



**Fig. 11.14**

**Example 11.44** *Find the Laplace transform of a rectangular pulse shown in Fig. 11.15.*

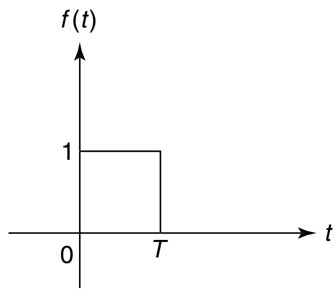



**Fig. 11.15**

**Solution** The rectangular pulse can be constructed from two functions as shown in Fig. 11.16. This function is known as gate function.

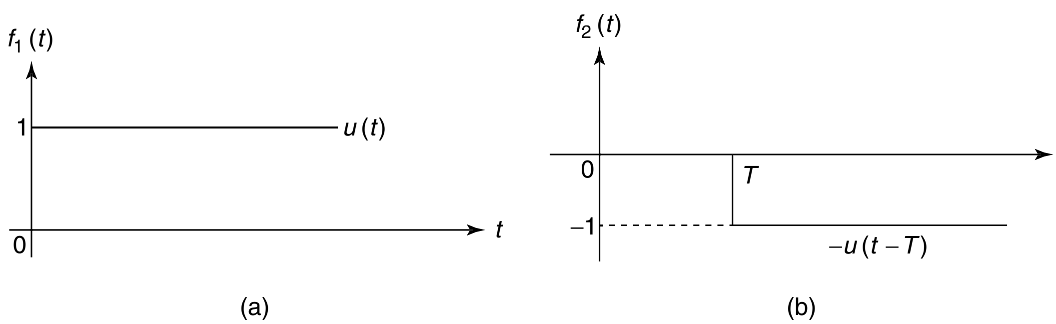



**Fig. 11.16**

$$f(t) = f_1(t) + f_2(t) = u(t) - u(t-T)$$

$$F(s) = L\{u(t)\} - L\{u(t-T)\} = \frac{1}{s} - \frac{1}{s}e^{-Ts} = \frac{1}{s}(1 - e^{-Ts})$$

**Example 11.45** *Find the Laplace transform of a sawtooth waveform shown in Fig. 11.17.*

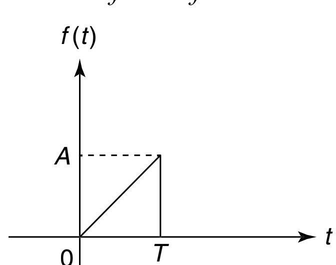



**Fig. 11.17**

**Solution** The sawtooth waveform can be constructed from three functions as shown in Fig. 11.18.

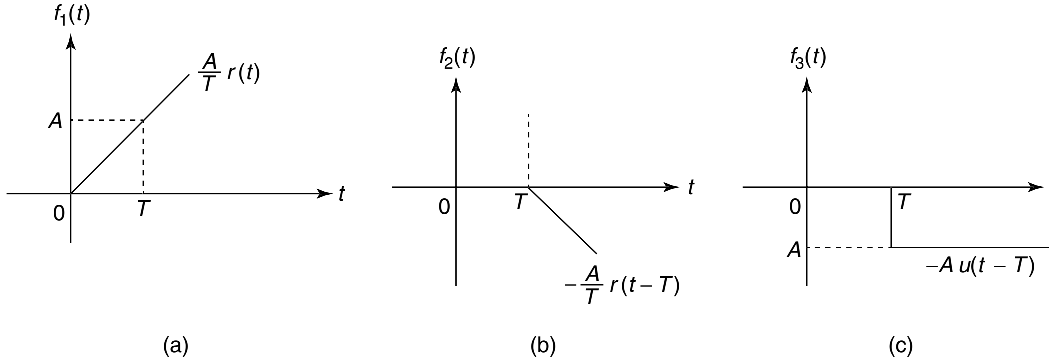



**Fig. 11.18**

$$f(t) = f_1(t) + f_2(t) + f_3(t) = \frac{A}{T} r(t) - \frac{A}{T} r(t-T) - Au(t-T)$$

$$F(s) = \frac{A}{T} L\{r(t)\} - \frac{A}{T} L\{r(t-T)\} - AL\{u(t-T)\} = \frac{A}{Ts^2} - \frac{A}{T}\frac{1}{s^2} e^{-Ts} - \frac{A}{s} e^{-Ts}$$

**Example 11.46** *Find the Laplace transform of a triangular waveform shown in Fig. 11.19.*

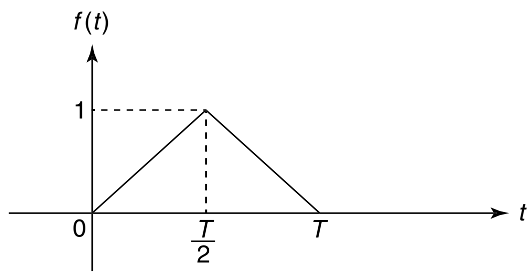



**Fig. 11.19**

**Solution** The triangular waveform can be constructed from three ramp functions as shown in Fig. 11.20.

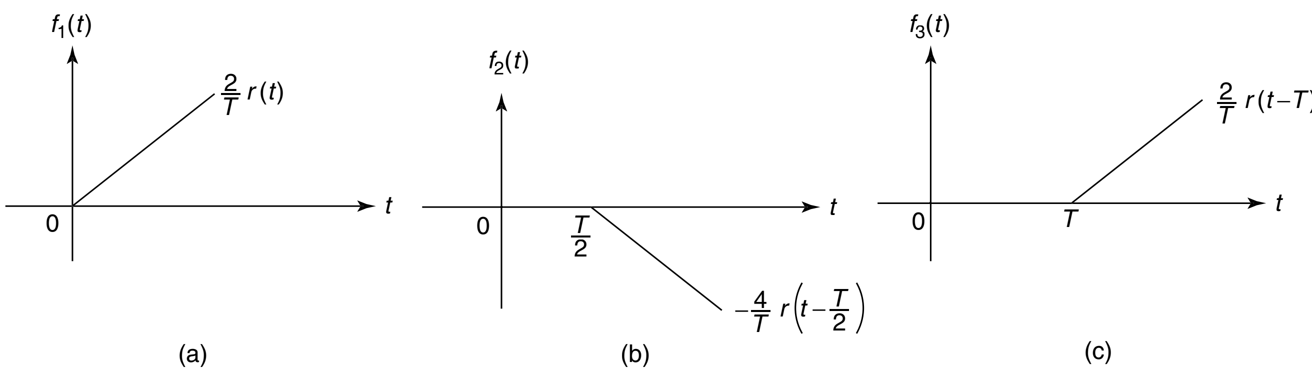



**Fig. 11.20**

$$f(t) = f_1(t) + f_2(t) + f_3(t) = \frac{2}{T} r(t) - \frac{4}{T} r\left(t - \frac{T}{2}\right) + \frac{2}{T} r(t-T)$$

$$F(s) = \frac{2}{T} L\{r(t)\} - \frac{4}{T} L\left\{r\left(t - \frac{T}{2}\right)\right\} + \frac{2}{T} L\{r(t-T)\} = \frac{2}{Ts^2} - \frac{4}{Ts^2} e^{-\frac{Ts}{2}} + \frac{2}{Ts^2} e^{-Ts}$$

**Example 11.47** *Find the Laplace transform of a trapezoidal pulse shown in Fig. 11.21.*

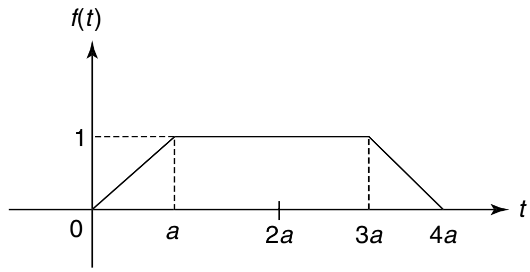



**Fig. 11.21**

**Solution** The trapezoidal waveform can be constructed from four ramp functions as shown in Fig.11.22.

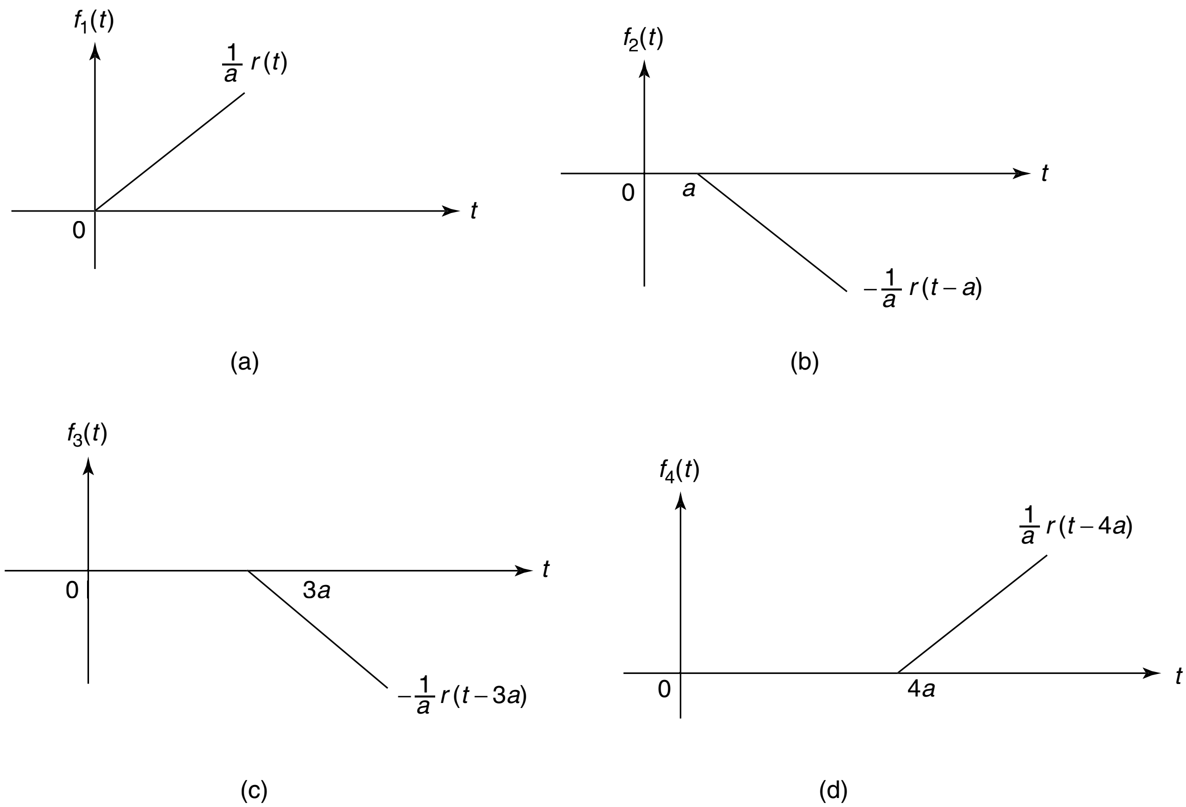



**Fig. 11.22**

$$f(t) = f_1(t) + f_2(t) + f_3(t) + f_4(t) = \frac{1}{a}r(t) - \frac{1}{a}r(t-a) - \frac{1}{a}r(t-3a) + \frac{1}{a}r(t-4a)$$

$$F(s) = \frac{1}{a}L\{r(t)\} - \frac{1}{a}L\{r(t-a)\} - \frac{1}{a}L\{r(t-3a)\} + \frac{1}{a}L\{r(t-4a)\} = \frac{1}{a}\frac{1}{s^2} - \frac{1}{a}\frac{1}{s^2}e^{-as} - \frac{1}{a}\frac{1}{s^2}e^{-3as} + \frac{1}{a}\frac{1}{s^2}e^{-4s}$$

$$= \frac{1}{as^2}(1 - e^{-as} - e^{-3as} + e^{-4as})$$

**Example 11.48** *Find the Laplace transform of a sinusoidal waveform shown in Fig. 11.23.*

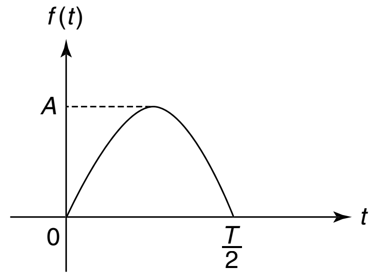



**Fig. 11.23**

**Solution** The waveform can be constructed from two functions as shown in Fig. 11.24.

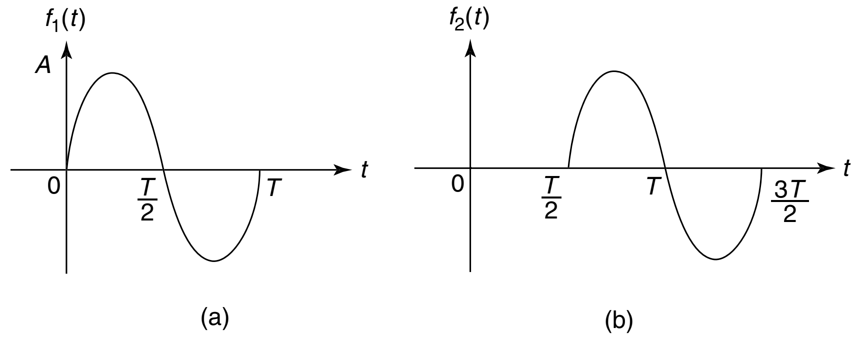



**Fig. 11.24**

$$f(t) = f_1(t) + f_2(t) = A\sin\omega t\; u(t) + A\sin\omega t\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)$$

where $$\omega = \frac{2\pi}{T}$$

$$F(s) = A\,L\{\sin\omega t\; u(t)\} + AL\left\{\sin\omega t\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)\right\} = \frac{A\omega}{s^2+\omega^2} + \frac{A\omega}{s^2+\omega^2}e^{-\frac{Ts}{2}} = \frac{A\omega}{s^2+\omega^2}\left(1+e^{-\frac{Ts}{2}}\right)$$

**Example 11.49** *Find the Laplace transform of the waveform shown in Fig. 11.25.*

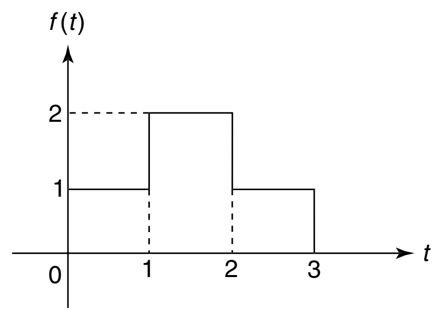



**Fig. 11.25**

**Solution** The function $f(t)$ can be expressed as sum of four step functions.

$$f(t) = u(t) + u(t-1) - u(t-2) - u(t-3)$$

$$F(s) = L\{u(t)\} + L\{u(t-1)\} - L\{u(t-2)\} - L\{u(t-3)\} = \frac{1}{s} + \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-3s} = \frac{1}{s}\left(1+e^{-s} - e^{-2s} - e^{-3s}\right)$$

**Example 11.50** *Determine the Laplace transform of the waveform shown in Fig. 11.26.*

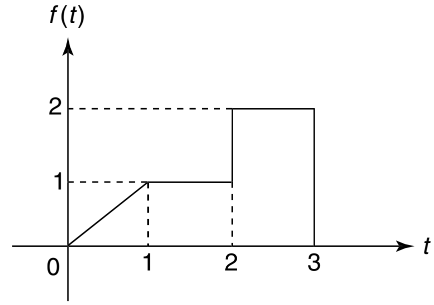



**Fig. 11.26**

**Solution** The function $f(t)$ can be expressed as sum of four functions as shown in Fig. 11.27.

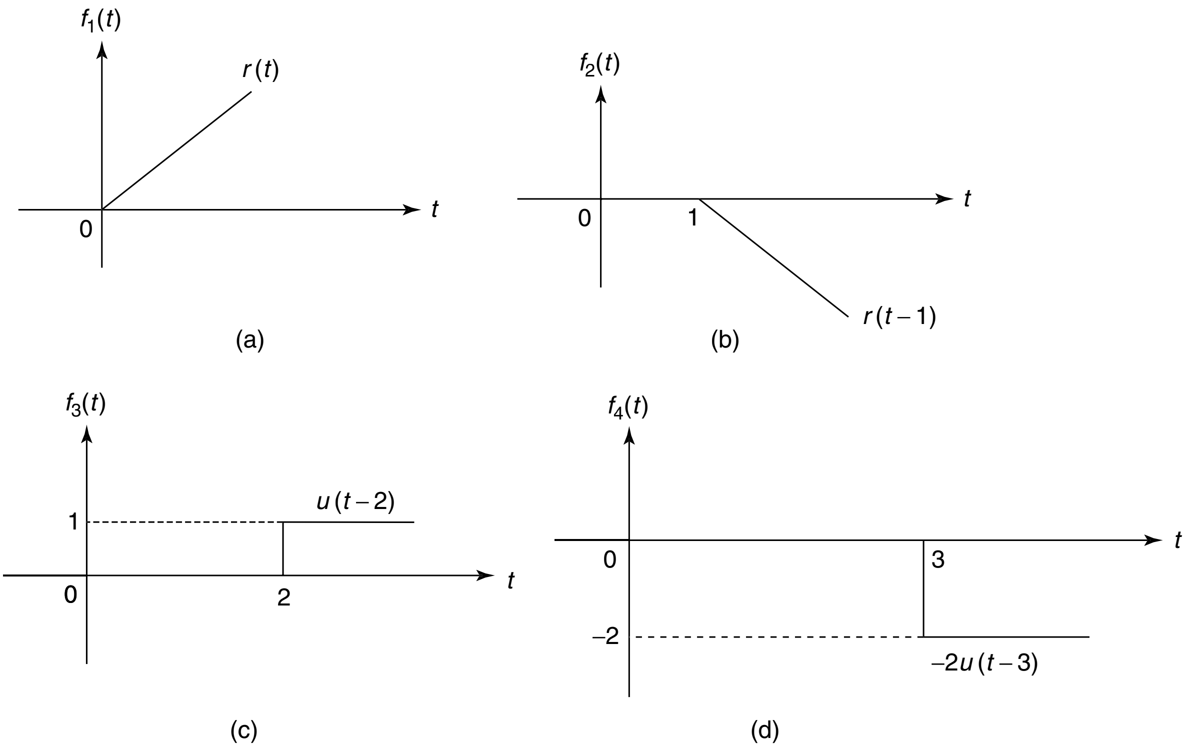



**Fig. 11.27**

$$f(t) = f_1(t) + f_2(t) + f_3(t) + f_4(t) = r(t) + r(t-1) + u(t-2) - 2u(t-3)$$

$$F(s) = L\{r(t)\} + L\{r(t-1)\} + L\{u(t-2)\} - 2L\{u(t-3)\} = \frac{1}{s^2} + \frac{1}{s^2}e^{-s} + \frac{1}{s}e^{-2s} - \frac{2}{s}e^{-3s}$$

**Example 11.51** *Find the Laplace transform of the waveform shown in Fig. 11.28.*

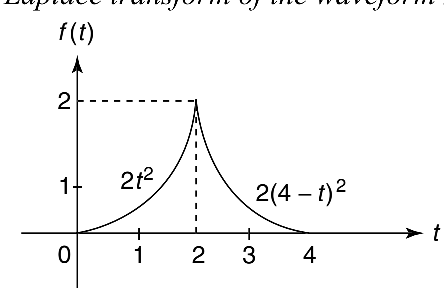



**Fig. 11.28**

**Solution** The given parabolic waveform can be constructed from three functions as shown in Fig.11.29.

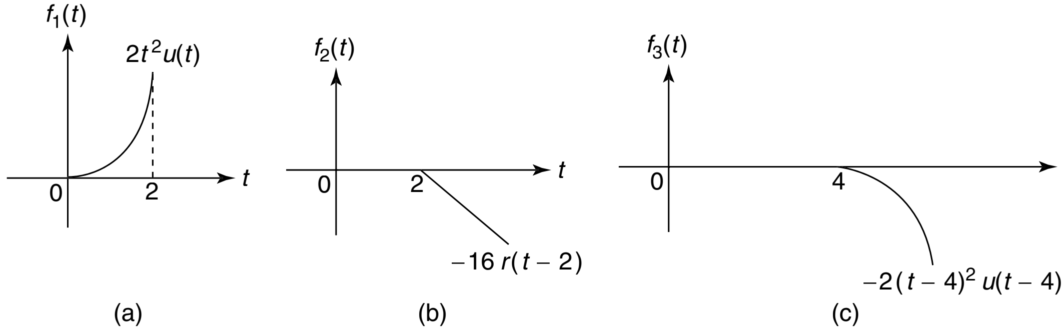



**Fig. 11.29**

$$F(t) = f_1(t) + f_2(t) + f_3(t) = 2t^2u(t) - 16r(t-2) - 2(t-4)^2u(t-4)$$

$$F(s) = 2L\{t^2u(t)\} - 16L\{r(t-2)\} - 2L\{(t-4)^2u(t-4)\} = 2\frac{2}{s^3} - 16\frac{1}{s^2}e^{-2s} - 2\frac{2}{s^3}e^{-4s} = \frac{4}{s^3}(1-e^{-4s}) - \frac{16}{s^2}e^{-2s}$$

**Example 11.52** *Find the Laplace transform of the waveform as shown in Fig. 11.30.*

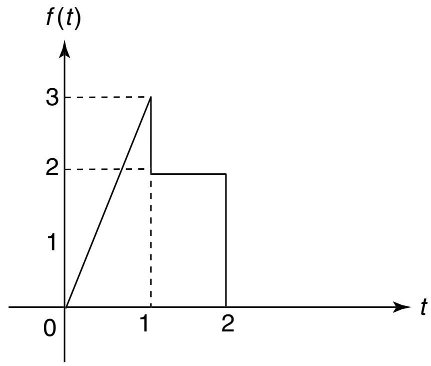



**Fig. 11.30**

**Solution** The given waveform can be constructed from four functions as shown in Fig. 11.31.

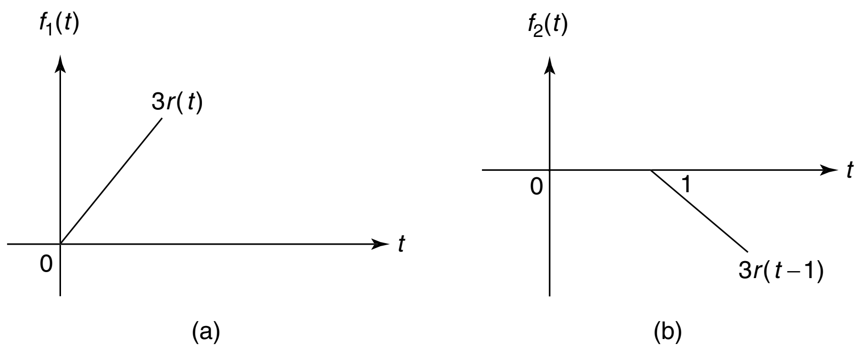



**Fig. 11.31**

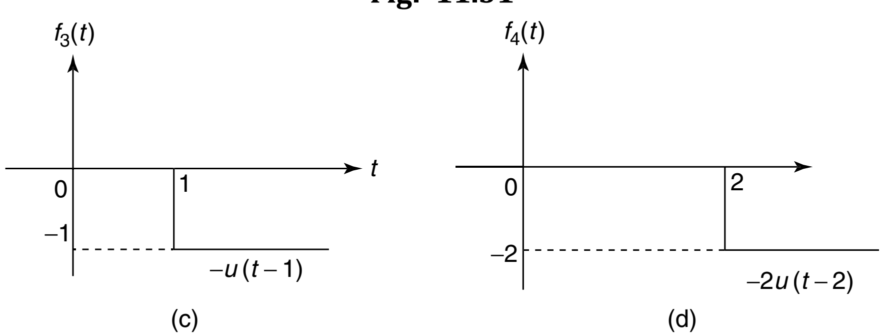



**Fig. 11.32**

$$f(t) = f_1(t) + f_2(t) + f_3(t) + f_4(t) = 3r(t) - 3r(t-1) - u(t-1) - 2u(t-2)$$

$$F(s) = 3L\{r(t)\} - 3L\{r(t-1)\} - L\{u(t-1)\} - 2L\{u(t-2)\} = \frac{3}{s^2} - \frac{3}{s^2}e^{-s} - \frac{1}{s}e^{-s} - \frac{2}{s}e^{-2s}$$

**Example 11.53** *Find the Laplace transform of the periodic waveform shown in Fig. 11.33.*

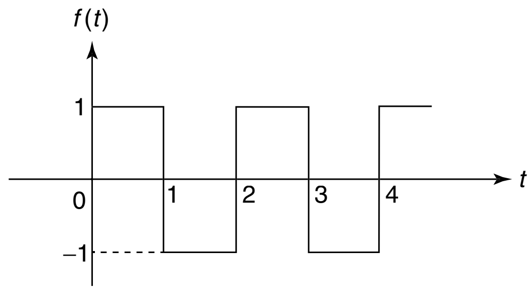



**Fig. 11.33**

**Solution** The function $f(t)$ is a periodic function with period 2.

The function $f_1(t)$ can be constructed from three functions by waveform synthesis.

$$f_1(t) = u(t) - 2u(t-1) + u(t-2)$$

$$F_1(s) = L\{u(t)\} - 2L\{u(t-1)\} + L\{u(t-2)\} = \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s}$$

The Laplace transform of periodic function $f(t)$ is

$$L\{f(t)\} = \frac{1}{1-e^{-2s}} F_1(s)$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s}\right)$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{1-2e^{-s}+e^{-2s}}{s}\right)$$

$$= \frac{(1-e^{-s})^2}{s(1-e^{-s})(1+e^{-s})}$$

$$= \frac{1}{s}\frac{1-e^{-s}}{1+e^{-s}}$$

$$= \frac{1}{s}\frac{e^{\frac{s}{2}} - e^{-\frac{s}{2}}}{e^{\frac{s}{2}} + e^{-\frac{s}{2}}}$$

$$= \frac{1}{s}\tanh\left(\frac{s}{2}\right)$$

**Example 11.54** *Find the Laplace transform of the waveform shown in Fig. 11.34.*

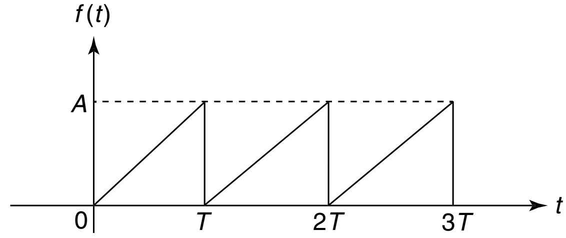



**Fig. 11.34**

**Solution** The function $f(t)$ is a periodic function with period $T$.

The function $f_1(t)$ can be constructed from three functions by waveform synthesis.

$$f_1(t) = \frac{A}{T}r(t) - \frac{A}{T}r(t-T) - Au(t-T)$$

$$F_1(s) = \frac{A}{T}L\{r(t)\} - \frac{A}{T}L\{r(t-T)\} - AL\{u(t-T)\} = \frac{A}{Ts^2} - \frac{A}{T}\frac{1}{s^2}e^{-Ts} - \frac{A}{s}e^{-Ts}$$

The Laplace transform of the periodic function $f(t)$ is

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}F_1(s) = \frac{1}{1-e^{-Ts}}\left(\frac{A}{Ts^2} - \frac{A}{T}e^{-Ts} - \frac{A}{s}e^{-Ts}\right)$$

**Example 11.55** *Find the Laplace transform of periodic waveform shown in Fig. 11.35.*

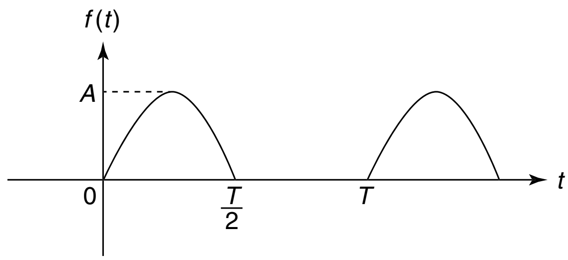



**Fig. 11.35**

**Solution** The function $f(t)$ is a periodic function with period $T$.

The function $f_1(t)$ can be constructed from two functions by waveform synthesis.

$$f_1(t) = A\sin\omega t\, u(t) + A\sin\omega t\left(t - \frac{T}{2}\right)u\left(t - \frac{T}{2}\right)$$

$$F_1(s) = AL\{\sin\omega t\, u(t)\} + AL\left\{\sin\omega t\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)\right\} = A\frac{\omega}{s^2+\omega^2} + A\frac{\omega}{s^2+\omega^2}e^{-\frac{Ts}{2}} = \frac{A\omega}{s^2+\omega^2}\left(1+e^{-\frac{Ts}{2}}\right)$$

The Laplace transform of the periodic function $f(t)$ is

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}F_1(s)$$

$$= \frac{1}{1-e^{-Ts}}\frac{A\omega}{s^2+\omega^2}\left(1+e^{-\frac{Ts}{2}}\right)$$

$$= \frac{A\omega}{s^2+\omega^2}\frac{1+e^{-\frac{Ts}{2}}}{\left(1-e^{-\frac{Ts}{2}}\right)\left(1+e^{-\frac{Ts}{2}}\right)}$$

$$= \frac{A\omega}{s^2+\omega^2}\frac{1}{1-e^{-\frac{Ts}{2}}}$$

## 11.7 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called inverse Laplace transform of $F(s)$ and symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where $L^{-1}$ is called the inverse Laplace transform operator.

Inverse Laplace transform can be found by the following methods:

(i) Standard results
(ii) Partial fraction expansion
(iii) Convolution theorem

### 11.7.1 Standard Results

Inverse Laplace transforms of some simple functions can be found by standard results and properties of Laplace transform.

**Example 11.56** *Find the inverse Laplace transform of* $\frac{s^2 - 3s + 4}{s^3}$.

**Solution**

$$F(s) = \frac{s^2 - 3s + 4}{s^3} = \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}$$

$$L^{-1}\{F(s)\} = 1 - 3t + 2t^2$$

**Example 11.57** *Find the inverse Laplace transform of* $\frac{3s+4}{s^2+9}$.

**Solution**

$$F(s) = \frac{3s+4}{s^2+9} = \frac{3s}{s^2+9} + \frac{4}{s^2+9}$$

$$L^{-1}\{F(s)\} = 3\cos 3t + \frac{4}{3}\sin 3t$$

**Example 11.58** *Find the inverse Laplace transform of* $\frac{4s+15}{16s^2-25}$.

**Solution**

$$F(s) = \frac{4s+15}{16s^2-25} = \frac{4s+15}{16\left(s^2 - \frac{25}{16}\right)} = \frac{1}{4}\frac{s}{s^2 - \frac{25}{16}} + \frac{15}{16}\frac{1}{s^2 - \frac{25}{16}}$$

$$L^{-1}\{F(s)\} = \frac{1}{4}\cosh\frac{5}{4}t + \frac{3}{4}\sinh\frac{5}{4}t$$

**Example 11.59** *Find the inverse Laplace transform of* $\frac{2s+2}{s^2+2s+10}$.

**Solution**

$$F(s) = \frac{2s+2}{s^2+2s+10} = \frac{2(s+1)}{(s+1)^2+9}$$

$$L^{-1}\{F(s)\} = 2e^{-t}L^{-1}\left\{\frac{s}{s^2+9}\right\} = 2e^{-t}\cos 3t$$

**Example 11.60** *Find the inverse Laplace transform of* $\dfrac{3s+7}{s^2-2s-3}$.

**Solution**

$$F(s)=\frac{3s+7}{s^2-2s-3}=\frac{3(s-1)+10}{(s-1)^2-4}=3\frac{(s-1)}{(s-1)^2-4}+10\frac{1}{(s-1)^2-4}$$

$$L^{-1}\{F(s)\}=3e^tL^{-1}\left\{\frac{s}{s^2-4}\right\}+10e^tL^{-1}\left\{\frac{1}{s^2-4}\right\}=3e^t\cosh 2t+5e^t\sinh 2t$$

### 11.7.2 Partial Fraction Expansion

Any function $F(s)$ can be written as $\dfrac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials in $s$. For performing partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

***Case I*** Factors are linear and distinct,

$$F(s)=\frac{P(s)}{(s+a)(s+b)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+a}+\frac{B}{s+b}$$

***Case II*** Factors are linear and repeated,

$$F(s)=\frac{P(s)}{(s+a)(s+b)^n}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+a}+\frac{B_1}{s+b}+\frac{B_2}{(s+b)^2}+\ldots+\frac{B_n}{(s+b)^n}$$

***Case III*** Factors are quadratic and distinct,

$$F(s)=\frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial-fraction expansion,

$$F(s)=\frac{As+B}{s^2+as+b}+\frac{Cs+D}{s^2+cs+d}$$

***Case IV*** Factors are quadratic are repeated,

$$F(s)=\frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial-fraction expansion,

$$F(s)=\frac{As+B}{s^2+as+b}+\frac{C_1s+D_1}{s^2+cs+d}+\frac{C_2s+D_2}{(s^2+cs+d)^2}+\ldots+\frac{C_ns+D_n}{(s^2+cs+d)^n}$$

**Example 11.61** *Find the inverse Laplace transform of* $\dfrac{s+2}{s(s+1)(s+3)}$.

**Solution**

$$F(s)=\frac{s+2}{s(s+1)(s+3)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+3}$$

$$A=sF(s)\big|_{s=0}=\frac{s+2}{(s+1)(s+3)}\bigg|_{s=0}=\frac{2}{3}$$

$$B=(s+1)F(s)\big|_{s=-1}=\frac{s+2}{s(s+3)}\bigg|_{s=-1}=-\frac{1}{2}$$

$$C=(s+3)F(s)\big|_{s=-3}=\frac{s+2}{s(s+1)}\bigg|_{s=-3}=-\frac{1}{6}$$

$$F(s)=\frac{2}{3}\cdot\frac{1}{s}-\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\}=\frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\}=\frac{2}{3}-\frac{1}{2}e^{-t}-\frac{1}{6}e^{-3t}$$

**Example 11.62** *Find the inverse Laplace transform of* $\dfrac{s+2}{s^2(s+3)}$.

**Solution**

$$F(s)=\frac{s+2}{s^2(s+3)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+3}$$

$$s+2=As(s+3)+B(s+3)+Cs^2$$
$$=As^2+3As+Bs+3B+Cs^2$$
$$=(A+C)s^2+(3A+B)s+3B$$

Comparing coefficients of $s^2, s^1$ and $s^0$,

$$A+C=0$$
$$3A+B=1$$
$$3B=2$$

Solving these equations,

$$A=\frac{1}{9},\quad B=\frac{2}{3},\quad C=-\frac{1}{9}$$

$$F(s)=\frac{1}{9}\cdot\frac{1}{s}+\frac{2}{3}\cdot\frac{1}{s^2}-\frac{1}{9}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\}=\frac{1}{9}L^{-1}\left\{\frac{1}{s}\right\}+\frac{2}{3}L^{-1}\left\{\frac{1}{s^2}\right\}-\frac{1}{9}L^{-1}\left\{\frac{1}{s+3}\right\}=\frac{1}{9}+\frac{2}{3}t-\frac{1}{9}e^{-3t}$$

**Example 11.63** *Find the inverse Laplace transform of* $\frac{s^2-15s-11}{(s+1)(s-2)^2}$.

**Solution**

$$F(s)=\frac{5s^2-15s-11}{(s+1)(s-2)^2}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$\begin{aligned}5s^2-15s-11&=A(s-2)^2+B(s+1)(s-2)+C(s+1)\\&=A(s^2-4s+4)+B(s^2-s-2)+C(s+1)\\&=As^2-4As+4A+Bs^2-Bs-2B+Cs+c\\&=(A+B)s^2-(4A+B-C)s+(4A-2B+C)\end{aligned}$$

Comparing coefficients of $s^2$, $s^1$ and $s^0$,

$$\begin{aligned}A+B&=5\\4A+B-C&=15\\4A-2B+C&=-11\end{aligned}$$

Solving these equations,

$$\begin{aligned}A&=1\\B&=4\\C&=-7\end{aligned}$$

$$F(s)=\frac{1}{s+1}+\frac{4}{s-2}-\frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{s+1}\right\}+4L^{-1}\left\{\frac{1}{s-2}\right\}-7L^{-1}\left\{\frac{1}{(s-2)^2}\right\}=e^{-t}+4e^{2t}-7te^{2t}$$

**Example 11.64** *Find the inverse Laplace transform of* $\frac{3s+1}{(s+1)(s^2+2)}$.

**Solution**

$$F(s)=\frac{3s+1}{(s+1)(s^2+2)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2+2}$$

$$\begin{aligned}3s+1&=A(s^2+2)+(Bs+C)(s+1)\\&=As^2+2A+Bs^2+Bs+Cs+C\\&=(A+B)s^2+(B+C)s+(2A+C)\end{aligned}$$

Comparing coefficients of $s^2, s^1$ and $s^0$,

$$A + B = 0$$
$$B + C = 3$$
$$2A + C = 1$$

Solving these equations,

$$A = -\frac{2}{3},\ B = \frac{2}{3},\ C = \frac{7}{3}$$

$$F(s) = -\frac{2}{3}\cdot\frac{1}{s+1} + \frac{2}{3}\cdot\frac{s}{s^2+2} + \frac{7}{3}\cdot\frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{7}{3}L^{-1}\left\{\frac{1}{s^2+2}\right\} = -\frac{2}{3}e^{-t} + \frac{2}{3}\cos\sqrt{2t} + \frac{7}{3\sqrt{2}}\sin\sqrt{2t}$$

**Example 11.65** *Find the inverse Laplace transform of* $\dfrac{s}{(s^2+1)(s^2+4)}$.

**Solution**

$$F(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\}\right] = \frac{1}{3}[\cos t - \cos 2t]$$

### 11.7.3 Convolution Theorem

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then $L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)\,du$

where $\int_0^t f_1(u) f_2(t-u)\,du = f_1(t) * f_2(t)$

***Proof***

$$F_1(s)\cdot F_2(s) = L\{f_1(t)\}\cdot L\{f_2(t)\} = \int_0^\infty e^{-su} f_1(u)\,du \cdot \int_0^\infty e^{-sv} f_2(v)\,dv = \int_0^\infty\int_0^\infty e^{-s(u+v)} f_1(u) f_2(v)\,du\,dv$$

$$= \int_0^\infty f_1(u)\left[\int_0^\infty e^{-s(u+v)} f_2(v)\,dv\right]du$$

Putting $u + v = t,\quad dv = dt$

When $\quad v = 0,\quad t = u$

$\quad v \to \infty,\quad t \to \infty$

$$F_1(s)\cdot F_2(s) = \int_0^\infty f_1(u)\left[\int_u^\infty e^{-st} f_2(t-u)\,dt\right]du = \int_0^\infty\int_0^\infty e^{-st} f_1(u) f_2(t-u)\,dt\,du$$

The region of integration is bounded by the lines $u = 0$ and $u = t$. To change the order of integration, draw a vertical strip which starts from line $u = 0$ and terminates on the line $u = t$. Hence, $u$ varies from 0 to $t$ and $t$ varies from 0 to $\infty$.

$$F_1(s)\cdot F_2(s) = \int_0^\infty e^{-st}\int_0^t f_1(u) f_2(t-u)\,du\,dt = L\left\{\int_0^t f_1(u) f_2(t-u)\,du\right\}$$

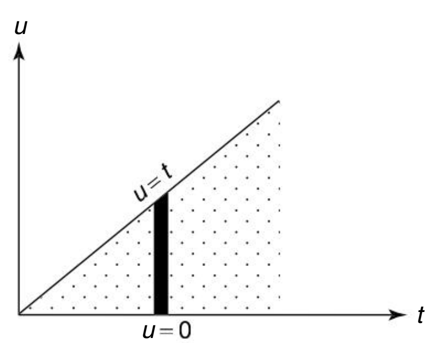



**Fig. 11.36**

Hence, $$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)du$$

**Note** Convolution operation is commutative, i.e.,

$$L\left\{\int_0^t f_1(u) f_2(t-u)du\right\} = L\left\{\int_0^t f_1(t-u) f_2(u)du\right\}$$

**Example 11.66** *Find the inverse Laplace transform of* $\dfrac{1}{(s+2)(s-1)}$.

**Solution**

$$F(s) = \frac{1}{(s+2)(s-1)}$$

Let

$$F_1(s) = \frac{1}{s+2} \qquad F_2(s) = \frac{1}{s-1}$$

$$f_1(t) = e^{-2t} \qquad f_2(t) = e^t$$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u} e^{t-u} du = e^t \int_0^t e^{-3u} du = e^t \left[\frac{e^{-3u}}{-3}\right]_0^t = \frac{e^t}{3}(1-e^{-3t})$$

**Example 11.67** *Find the inverse Laplace transform of* $\dfrac{1}{s^2(s+1)^2}$.

**Solution**

$$F(s) = \frac{1}{s^2(s+1)^2}$$

Let

$$F(s) = \frac{1}{(s+1)^2} \qquad F_2(s) = \frac{1}{s^2}$$

$$f_1(t) = te^{-t} \qquad f_2(t) = t$$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t ue^{-u}(t-u)\,du = \int_0^t (ut-u^2)e^{-u}\,du = \left[(ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\right]_0^t$$

$$= te^{-t} + 2e^{-t} + t - 2$$

**Example 11.68** *Find the inverse Laplace transform of* $\dfrac{1}{(s-2)(s+2)^2}$.

**Solution**

$$F(s) = \frac{1}{(s-2)(s+2)^2}$$

Let

$$F_1(s) = \frac{1}{(s+2)^2} \qquad F_2(s) = \frac{1}{s-2}$$

$$f_1(t) = te^{-2t} \qquad f_2(t) = e^{2t}$$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t ue^{-2u}\, e^{2(t-u)} \int_0^t du = e^{2t} \int_0^t ue - 4u\,du = e^{2t}\left[\frac{ue^{-4u}}{-4} - \frac{e^{-4u}}{16}\right]_0^t = e^{2t}\left(\frac{-te^{-4t}}{4} - \frac{e^{-4t}}{16} + \frac{1}{16}\right)$$

$$= \frac{e^{2t}}{16} - \frac{te^{-2t}}{4} - \frac{e^{-2t}}{16} = \frac{1}{16}\left(e^{2t} - e^{-2t} - 4t\,e^{-2t}\right)$$

**Example 11.69** *Find the inverse Laplace transform of* $\dfrac{1}{s^2(s^2+1)}$.

**Solution**

$$F(s)=\frac{1}{s^2(s^2+1)}$$

Let

$$F_1(s)=\frac{1}{s^2+1} \qquad F_2(s)=\frac{1}{s^2}$$

$$f_1(t)=\sin t \qquad f_2(t)=t$$

By convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t \sin u\,(t-u)\,du=\left[(t-u)(-\cos u)-\sin u\right]_0^t=t-\sin t$$

**Example 11.70** *Find the inverse Laplace transform of* $\dfrac{1}{(s+1)(s^2+1)}$.

**Solution**

$$F(s)=\frac{1}{(s+1)(s^2+1)}$$

Let

$$F_1(s)=\frac{1}{s^2+1} \qquad F_2(s)=\frac{1}{s+1}$$

$$f_1(t)=\sin t \qquad f_2(t)=e^{-t}$$

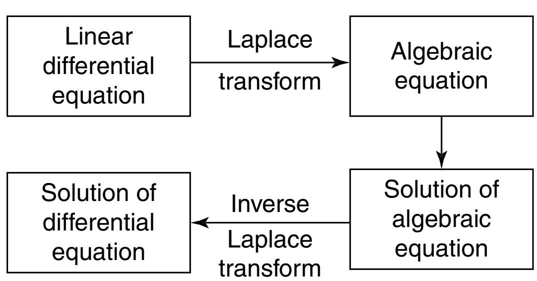
By convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t \sin u\; e^{-(t-u)}\;du=\int_1^t e^{u-t}\sin u\,du=e^{-t}\left[\frac{e^u}{2}(\sin u-\cos u)\right]_0^t=\frac{e^{-t}}{2}[e^t(\sin t-\cos t)+1]$$

$$=\frac{1}{2}(\sin t-\cos t)+\frac{1}{2}e^{-t}$$

## 11.8 SOLUTION OF DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform is useful in solving linear differential equations with given initial conditions by using algebraic methods. Initial conditions are included from the very beginning of the solution.



**Example 11.71** *Solve* $\frac{dy}{dt} + 2y = e^{-3t}$, $y(0) = 1$.

**Solution** Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s+3}$$

$$sY(s) - 1 + 2Y(s) = \frac{1}{s+3} \qquad [\because y(0) = 1]$$

$$(s+2)Y(s) = \frac{1}{s+3} + 1 = \frac{s+4}{s+3}$$

$$Y(s) = \frac{s+4}{(s+2)(s+3)}$$

By partial-fraction expansion,

$$Y(s) = \frac{A}{s+2} + \frac{B}{s+3}$$

$$A = (s+2)Y(s)\big|_{s=-2} = \left.\frac{s+4}{s+3}\right|_{s=-2} = 2$$

$$B = (s+3)Y(s)\big|_{s=-3} = \left.\frac{s+4}{s+2}\right|_{s=-3} = -1$$

$$Y(s) = \frac{2}{s+2} - \frac{1}{s+3}$$

Taking inverse Laplace transform of both the sides.

$$y(t) = 2e^{-2t} - e^{-3t}$$

**Example 11.72** *Solve* $y'' + y = t$, $y(0) = 1$, $y'(0) = 0$.

**Solution** Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) - s + Y(s) = \frac{1}{s^2} \qquad [\because y(0) = 1, y'(0) = 0]$$

$$(s^2+1)Y(s) = \frac{1}{s^2} + s = \frac{s^3+1}{s^2}$$

$$Y(s) = \frac{s^3+1}{s^2(s^2+1)} = \frac{s}{s^2+1} + \frac{1}{s^2(s^2+1)} = \frac{s}{s^2+1} + \frac{s^2+1-s^2}{s^2(s^2+1)} = \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \cos t + t - \sin t$$

**Example 11.73** *Solve* $y'' + y = t^2 + 2t$, $y(0) = 4$, $y'(0) = -2$.

**Solution** Taking Laplace transform of both sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2Y(s) - 4s + 2 + sY(s) - 4 = \frac{2}{s^3} + \frac{2}{s^2}$$

$$(s^2+s)\,Y(s)=\frac{2}{s^3}+\frac{2}{s^2}+4s+2=\frac{2(1+s)}{s^3}+4s+2$$

$$Y(s)=\frac{2(1+s)}{s^3(s^2+s)}+\frac{4s}{s^2+s}+\frac{2}{s^2+s}=\frac{2}{s^4}+\frac{4}{s+1}+\frac{2}{s}-\frac{2}{s+1}=\frac{2}{s^4}+\frac{2}{s}-\frac{2}{s+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{t^3}{3}+2+2e^{-t}$$

**Example 11.74** *Solve* $y''+4y=\delta(t),\ y(0)=0,\ y'(0)=0.$

**Solution** Taking Laplace transform of both the sides,

$$s^2Y(s)-sy(0)-y'(0)+4Y(s)=1$$

$$s^2\,Y(s)+4\,Y(s)=1 \qquad [\because y(0)=0,\ y'(0)=0]$$

$$(s^2+4)\,Y(s)=1$$

$$Y(s)=\frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{1}{2}\sin 2t$$

**Example 11.75** *Solve* $y''+3y'+2y=t\delta(t-1),\ y(0)=0,\ y'(0)=0.$

**Solution** Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+3\left[sY(s)-y(0)\right]+2Y(s)=e^{-s}$$

$$s^2Y(s)+3s\,Y(s)+2Y(s)=e^{-s} \qquad [\because y(0)=0,\ y'(0)=0]$$

$$(s^2+3s+2)\,Y(s)=e^{-s}$$

$$Y(s)=\frac{e^{-s}}{s^2+3s+2}=\frac{e^{-s}}{(s+1)(s+2)}=e^{-s}\left(\frac{1}{s+1}-\frac{1}{s+2}\right)$$

Taking inverse Laplace transform of both the sides,

$$y(t)=e^{-(t-1)}u(t-1)-e^{-2(t-1)}u(t-1)$$

**Example 11.76** *Solve* $y''+4y=u(t-2),\ y(0)=0,\ y'(0)=1.$

**Solution** Taking Laplace transform of both the sides,

$$\left[s^2\,Y(s)-sy(0)-y'(0)\right]+4Y(s)=\frac{e^{-2s}}{s}$$

$$s^2\,Y(s)-1+4Y(s)=\frac{e^{-2s}}{s} \qquad [\because y(0)=0,\ y'(0)=1]$$

$$(s^2+4)Y(s)=\frac{e^{-2s}}{s}+1$$

$$Y(s)=\frac{e^{-2s}}{s(s^2+4)}+\frac{1}{s^2+4}=\frac{e^{-2s}}{4}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)+\frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{1}{4}u(t-2)-\frac{1}{4}\cos 2(t-2)u(t-2)+\frac{1}{2}\sin 2t$$

## 11.9 SOLUTION OF A SYSTEM OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

The Laplace transform can also be used to solve two or more simultaneous differential equations. The Laplace transform method transforms the differential equations into algebraic equations.

**Example 11.77** *Solve* $\frac{dx}{dt} + y = \sin t.$

$$\frac{dy}{dt} + x = \cos t$$

*where* $x(0) = 0$ *and* $y(0) = 2$.

**Solution** Taking Laplace transform of both the equations,

$$sX(s) - x(0) + Y(s) = \frac{1}{s^2+1}$$

$$sX(s) + Y(s) = \frac{1}{s^2+1} \qquad \ldots\text{(i)}$$

and

$$sY(s) - y(0) + X(s) = \frac{s}{s^2+1}$$

$$sY(s) + X(s) = \frac{s}{s^2+1} + 2$$

$$sY(s) + X(s) = \frac{2s^2+s+2}{s^2+1} \qquad \ldots\text{(ii)}$$

Multiplying Eq. (i) by $s$,

$$s^2X(s) + sY(s) = \frac{s}{s^2+1} \qquad \ldots\text{(iii)}$$

Subtracting Eq. (iii) from Eq. (ii),

$$(s^2-1)X(s) = -2$$

$$X(s) = -\frac{2}{s^2-1} \qquad \ldots\text{(iv)}$$

Substituting $X(s)$ in Eq. (i),

$$Y(s) = \frac{1}{s^2+1} + 2\frac{s}{s^2-1} \qquad \ldots\text{(v)}$$

Taking inverse Laplace transform of Eqs (iv) and (v),

$$x(t) = -2\sinh t$$

and

$$y(t) = \sin t + 2\cosh t$$

**Example 11.78** *Solve* $\frac{dx}{dt} - y = e^t$

$$\frac{dy}{dt} + x = \sin t$$

*where* $x(0) = 1$ *and* $y(0) = 0$.

**Solution** Taking Laplace transform of both the equations,

$$sX(s)-x(0)-Y(s)=\frac{1}{s-1}$$

$$sX(s)-Y(s)=\frac{1}{s-1}+1=\frac{s}{s-1} \qquad \ldots\text{(i)}$$

and $$sY(s)-y(0)+X(s)=\frac{1}{s^2+1}$$

$$sY(s)+X(s)=\frac{1}{s^2+1} \qquad \ldots\text{(ii)}$$

Multiplying Eq. (i) by $s$,

$$s^2X(s)-sY(s)=\frac{s^2}{s-1} \qquad \ldots\text{(iii)}$$

Adding Eqs (ii) and (iii),

$$(s^2+1)X(s)=\frac{1}{s^2+1}+\frac{s^2}{s-1}$$

$$X(s)=\frac{1}{(s^2+1)^2}+\frac{s^2}{(s-1)(s^2+1)}$$

$$=\frac{1}{(s^2+1)^2}+\frac{1}{2}\left(\frac{1}{s-1}+\frac{s}{s^2+1}+\frac{1}{s^2+1}\right) \qquad \ldots\text{(iv)}$$

Substituting $X(s)$ in Eq. (i),

$$Y(s)=sX(s)-\frac{s}{s-1}=\frac{s}{(s^2+1)}-\frac{s^3}{(s-1)(s^2+1)}-\frac{s}{s-1}$$

$$Y(s)=\frac{s}{(s^2+1)^2}-\frac{s}{(s-1)(s^2+1)}$$

$$=\frac{s}{(s^2+1)^2}-\frac{1}{2}\left(\frac{1}{s-1}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right) \qquad \ldots\text{(v)}$$

Taking the inverse Laplace transform of Eqs. (iv) and (v),

$$x(t)=\frac{1}{2}(\sin t-t\cos t)+\frac{1}{2}(e^t+\cos t+\sin t)=\frac{1}{2}(e^t+\cos t+2\sin t-t\cos t)$$

and $$y(t)=\frac{1}{2}t\sin t-\frac{1}{2}(e^t-\cos t+\sin t)=\frac{1}{2}(t\sin t-e^t+\cos t-\sin t)$$

**Example 11.79** *Solve* $$\frac{dx}{dt}+5x-2y=t$$

$$\frac{dy}{dt}+2x+y=0$$

where $x(0)=0$ *and* $y(0)=0$.

**Solution** Taking Laplace transform of both the equations,

$$s\,X(s) - x(0) + 5\,X(s) - 2Y(s) = \frac{1}{s^2}$$

$$(s+5)\,X(s) - 2Y(s) = \frac{1}{s^2} \qquad \ldots\text{(i)}$$

and

$$s\,Y(s) - y(0) + 2X(s) + Y(s) = 0$$

$$2X(s) + (s+1)\,Y(s) = 0 \qquad \ldots\text{(ii)}$$

Multiplying Eq. (i) by $\frac{1}{2}(s+1)$,

$$\frac{1}{2}(s+5)(s+1)X(s) - (s+1)Y(s) = \frac{s+1}{2s^2} \qquad \ldots\text{(iii)}$$

Adding Eqs (ii) and (iii),

$$X(s) = \frac{s+1}{s^2(s+3)^2} \qquad \ldots\text{(iv)}$$

Substituting $X(s)$ in Eq. (ii),

$$Y(s) = -\frac{2}{s^2(s+3)^2} \qquad \ldots\text{(v)}$$

Now,

$$X(s) = \frac{s+1}{s^2(s+3)^2}$$

By partial-fraction expansion,

$$X(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$s+1 = A\,s(s+3)^2 + B(s+3)^2 + C(s+3)s^2 + D\,s^2 \qquad \ldots\text{(vi)}$$

$$= As(s^2+6s+9) + B(s^2+6s+9) + C(s^3+3s^2) + Ds^2$$

$$= As^3 + 6As^2 + 9As + Bs^2 + 6Bs + 9B + Cs^3 + 3Cs^2 + Ds^2$$

$$= (A+C)s^3 + (6A+B+3C+D)s^2 + (9A+6B)s + 9B$$

Comparing coefficients of $s^3, s^2, s^1$ and $s^0$,

$$A + C = 0$$

$$6A + B + 3C + D = 0$$

$$9A + 6B = 1$$

$$9B = 1$$

Solving these equations,

$$A = \frac{1}{27},\ B = \frac{1}{9},\ C = -\frac{1}{27},\ D = -\frac{2}{9}$$

$$X(s) = \frac{1}{27}\cdot\frac{1}{s} + \frac{1}{9}\cdot\frac{1}{s^2} - \frac{1}{27}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{(s+3)^2}$$

Taking inverse Laplace transform of both the sides,

$$x(t) = \frac{1}{27} + \frac{1}{9}t - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

Similarly,

$$Y(s) = \frac{-2}{s^2(s+3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2} = \frac{4}{27}\cdot\frac{1}{s} - \frac{2}{9}\cdot\frac{1}{s^2} - \frac{4}{27}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{(s+3)^2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{4}{27} - \frac{2}{9}t - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

## 11.10 THE TRANSFORMED CIRCUIT

Voltage–current relationships of network elements can also be represented in the frequency domain.

***1. Resistor*** For the resistor, the *v*–*i* relationship in time domain is

$$v(t) = R\,i(t)$$

The corresponding frequency–domain relation are given as

$$V(s) = RI(s)$$

The transformed network is shown in Fig 11.37.

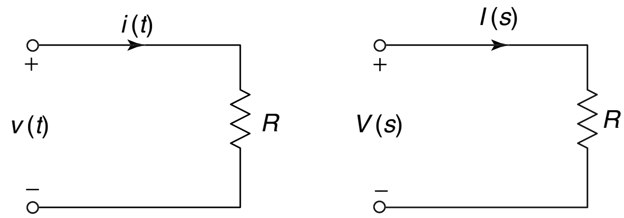



**Fig. 11.37** *Resistor*

***2. Inductor*** For the inductor, the *v*–*i* relationships in time domain are

$$v(t) = L\frac{di}{dt}$$

$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt + i(0)$$

The corresponding frequency-domain relation are given as

$$V(s) = Ls\,I(s) - L\,i(0)$$

$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0)}{s}$$

The transformed network is shown in Fig 11.38.

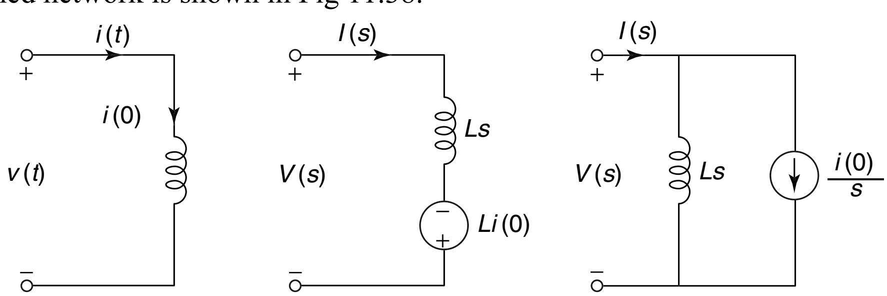



**Fig. 11.38** *Inductor*

**3. Capacitor** For capacitor, the *v*–*i* relationships in time domain are

$$v(t) = \frac{1}{C}\int_0^t i(t)\,dt + v(0)$$

$$i(t) = C\frac{dv}{dt}$$

The corresponding frequency–domain relations are given as

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s}$$

$$I(s) = CsV(s) - Cv(0)$$

The transformed network is shown in Fig 11.39.

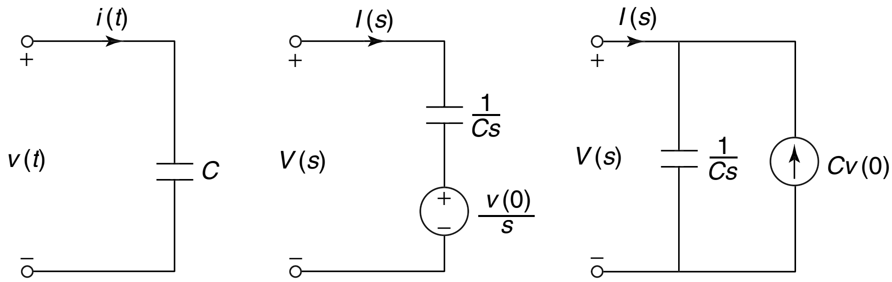



**Fig. 11.39** *Capacitor*

## 11.11 RESISTOR–INDUCTOR CIRCUIT

Consider a series *RL* circuit as shown in Fig. 11.40. The switch is closed at time $t = 0$.

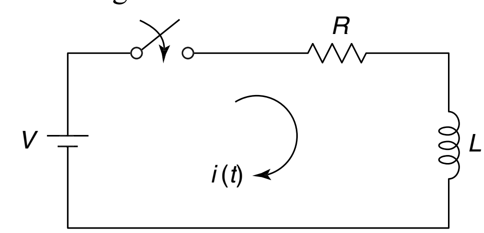



**Fig. 11.40** *RL circuit*

For $t > 0$, the transformed network is shown in Fig. 11.41.

Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - Ls\,I(s) = 0$$

$$I(s) = \frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)}$$

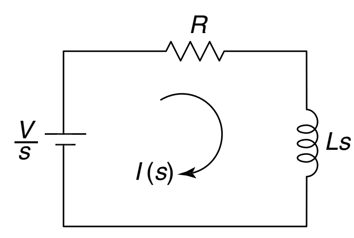



**Fig. 11.41** *Transformed network*

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s+\frac{R}{L}}$$

$$A = sI(s)\big|_{s=0} = s \times \left.\frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)}\right|_{s=0} = \frac{V}{R}$$

$$B = \left(s+\frac{R}{L}\right)I(s)\bigg|_{s=-\frac{R}{L}} = \left(s+\frac{R}{L}\right)\times\frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)}\Bigg|_{s=-\frac{R}{L}} = -\frac{V}{R}$$

$$I(s) = \frac{\frac{V}{R}}{s} + \frac{\left(-\frac{V}{R}\right)}{s+\frac{R}{L}}$$

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t}$$

$$= \frac{V}{R}\left[1-e^{-\frac{R}{L}t}\right] \quad \text{for } t > 0$$

**Example 11.80** *In the network of Fig. 11.42, the switch is moved from the position 1 to 2 at t = 0, steady-state condition having been established in the position 1. Determine i (t) for t > 0.*

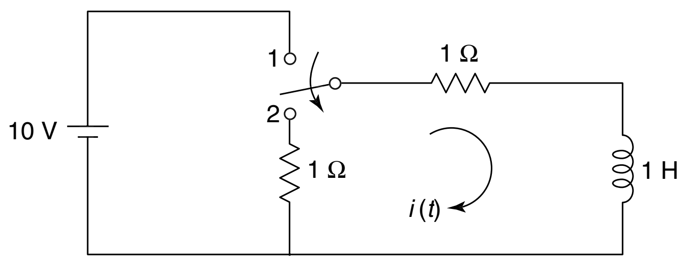



**Fig. 11.42**

**Solution** At $t = 0^-$, the network is shown in Fig 11.43. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{10}{1} = 10 \text{ A}$$

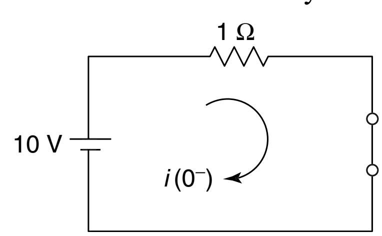



**Fig. 11.43**

Since the current through the inductor cannot change instantaneously,

$$i\,(0^+) = 10 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 11.44.

Applying KVL to the mesh for $t > 0$,

$$-I(s) - I(s) - sI(s) + 10 = 0$$

$$I(s)(s+2) = 10$$

$$I(s) = \frac{10}{s+2}$$

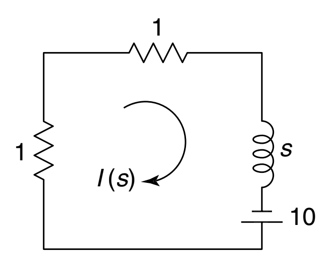



**Fig. 11.44**

Taking inverse Laplace transform,

$$i\,(\text{t}) = 10e^{-2t} \quad \text{for } t > 0$$

**Example 11.81** *The network of Fig. 11.45 was initially in the steady state with the switch in the position a. At t = 0, the switch goes from a to b. Find an expression for voltage v (t) for t > 0.*

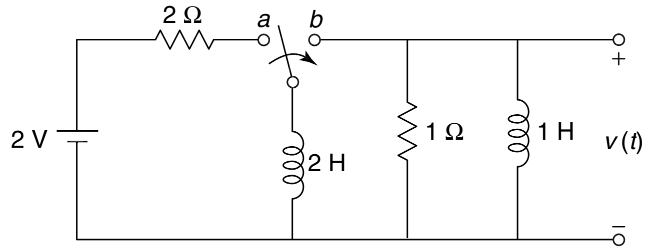



**Fig. 11.45**

**Solution** At $t = 0^-$, the network is shown in Fig 11.46. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor of 2H acts as a short circuit.

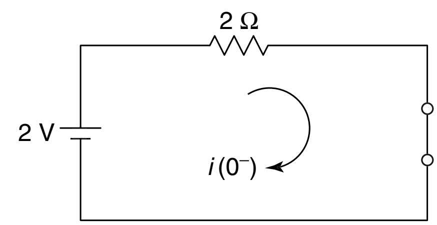



**Fig. 11.46**

$$i(0^-) = \frac{2}{2} = 1 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 11.47.

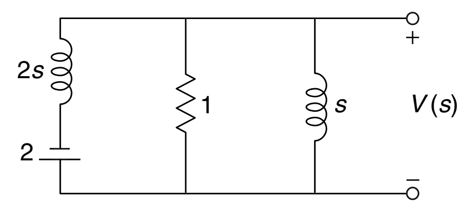



**Fig. 11.47**

Applying KCL at the node for $t > 0$,

$$\frac{V(s)+2}{2s} + \frac{V(s)}{1} + \frac{V(s)}{s} = 0$$

$$V(s)\left(1+\frac{3}{2s}\right) = -\frac{1}{s}$$

$$V(s) = \frac{-\frac{1}{s}}{\frac{2s+3}{2s}} = -\frac{2}{2s+3} = -\frac{1}{s+1.5}$$

Taking the inverse Laplace transform,

$$v(t) = -e^{-1.5t} \qquad \text{for } t > 0$$

**Example 11.82** *In the network of Fig. 11.48, the switch is opened at t = 0. Find i(t).*

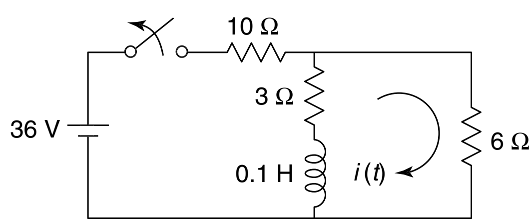



**Fig. 11.48**

**Solution** At $t = 0^-$, the network is shown in Fig. 11.49. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the inductor acts as a short circuit.

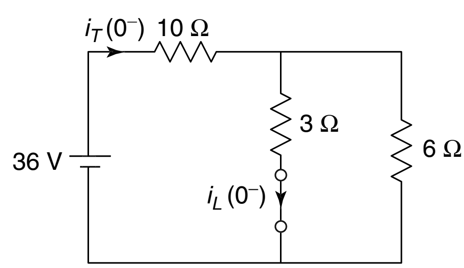



**Fig. 11.49**

$$i_T(0^-) = \frac{36}{10+(3\,||\,6)} = \frac{36}{10+2} = 3 \text{ A}$$

$$i_L(0^-) = 3\times\frac{6}{6+3} = 2 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 11.50.

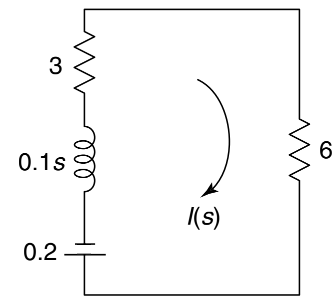



**Fig. 11.50**

Applying KVL to the mesh for $t > 0$,

$$-0.2 - 0.1s\, I(s) - 3I(s) - 6I(s) = 0$$

$$0.1sI(s) + 9I(s) = 0.2$$

$$I(s) = \frac{0.2}{0.1s + 9} = \frac{2}{s + 90}$$

Taking inverse Laplace transform,

$$i(t) = 2e^{-90\,t} \qquad \text{for } t > 0$$

**Example 11.83** *The network shown in Fig. 11.51 has acquired steady-state with the switch closed for t < 0. At t = 0, the switch is opened. Obtain i (t) for t > 0.*

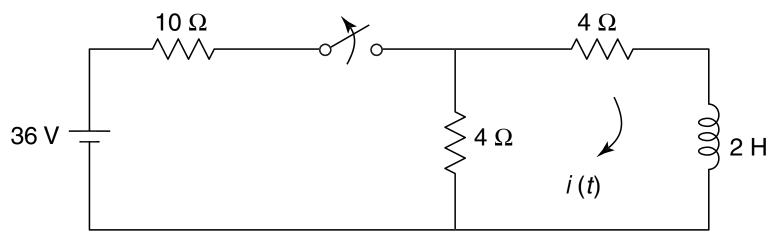



**Fig. 11.51**

**Solution** At $t = 0^-$, the network is shown in Fig 11.52. At $t = 0^-$, the switch is closed and the network has acquired steady-state. Hence, the inductor acts as a short circuit.

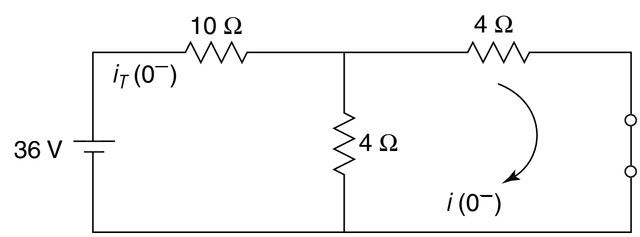



**Fig. 11.52**

$$i_T(0^-) = \frac{36}{10 + (4 \,||\, 4)} = \frac{36}{10 + 2} = 3 \text{ A}$$

$$i(0^-) = 3 \times \frac{4}{4 + 4} = 1.5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1.5 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 11.53.

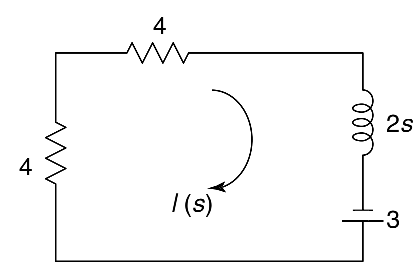



**Fig. 11.53**

Applying KVL to the mesh for $t > 0$,

$$-4I(s) - 4I(s) - 2sI(s) + 3 = 0$$

$$8I(s) + 2sI(s) = 3$$

$$I(s) = \frac{3}{2s + 8} = \frac{1.5}{s + 4}$$

Taking the inverse Laplace transform,

$$i(t) = 1.5e^{-4\,t} \qquad \text{for } t > 0$$

**Example 11.84** *In the network shown in Fig. 11.54, the switch is closed at t = 0, the steady-state being reached before t = 0. Determine current through inductor of 3 H.*

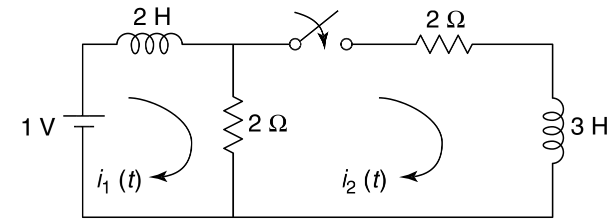



**Fig. 11.54**

**Solution** At $t = 0^-$, the network is shown in Fig. 11.55. At $t = 0^-$, steady-state condition is reached. Hence, the inductor of 2 H acts as a short circuit.

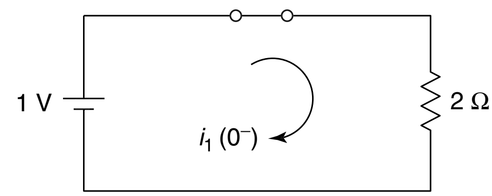



**Fig. 11.55**

$$i_1(0^-) = \frac{1}{2}\text{ A}$$

$$i_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i_1(0^+) = \frac{1}{2}\text{ A}$$

$$i_2(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 11.56.

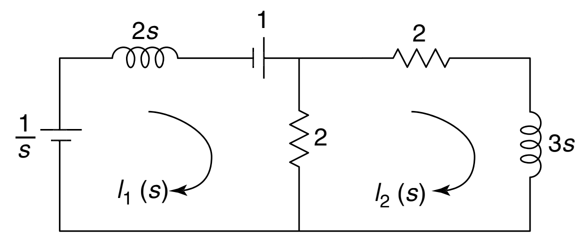



**Fig. 11.56**

Applying KVL to Mesh 1,

$$\frac{1}{s} - 2s\,I_1(s) + 1 - 2[I_1(s) - I_2(s)] = 0$$

$$(2+2s)\,I_1(s) - 2I_2(s) = 1 + \frac{1}{s}$$

Applying KVL to Mesh 2,

$$-2\,[I_2(s) - I_1(s)] - 2I_2(s) - 3s\,I_2(s) = 0$$

$$-2I_1(s) + (4+3s)\,I_2(s) = 0$$

By Cramer's rule,

$$I_2(s) = \frac{\begin{vmatrix} 2+2s & 1+\frac{1}{s} \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 2+2s & -2 \\ -2 & 4+3s \end{vmatrix}} = \frac{\frac{2}{s}(s+1)}{(2+2s)(4+3s)-4} = \frac{s+1}{s(3s^2+7s+2)} = \frac{s+1}{3s\left(s+\frac{1}{3}\right)(s+2)} = \frac{\frac{1}{3}(s+1)}{s(s+2)\left(s+\frac{1}{3}\right)}$$

By partial-fraction expansion,

$$I_2(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+\frac{1}{3}}$$

$$A = s\,I_2(s)\Big|_{s=0} = \left.\frac{\frac{1}{3}(s+1)}{(s+2)\left(s+\frac{1}{3}\right)}\right|_{s=0} = \frac{1}{2}$$

$$B = (s+2)I_2(s)\Big|_{S=-2} = \left.\frac{\frac{1}{3}(s+1)}{s\left(s+\frac{1}{3}\right)}\right|_{s=-2} = -\frac{1}{10}$$

$$C = \left(s+\frac{1}{3}\right) I_2(s)\Big|_{s=-\frac{1}{3}} = \left.\frac{\frac{1}{3}(s+1)}{s(s+2)}\right|_{s=-\frac{1}{3}} = -\frac{2}{5}$$

$$I_2(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{10}\frac{1}{s+2} - \frac{2}{5}\frac{1}{s+\frac{1}{3}}$$

Taking inverse Laplace transform

$$i_2(t) = \frac{1}{2} - \frac{1}{10}e^{-2t} - \frac{2}{5}e^{-\frac{1}{3}t} \quad \text{for } t > 0$$

**Example 11.85** *In the network of Fig. 11.57, the switch is closed at t = 0 with the network previously unenergised. Determine currents $i_1(t)$.*

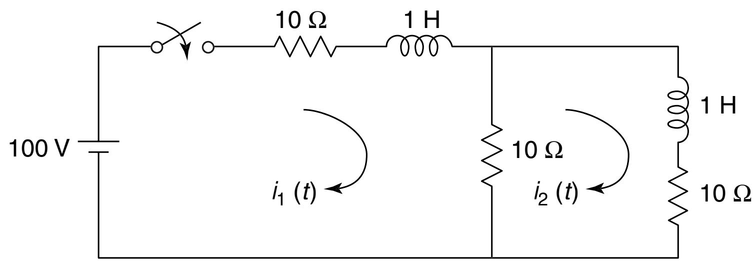



**Fig. 11.57**

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.58.

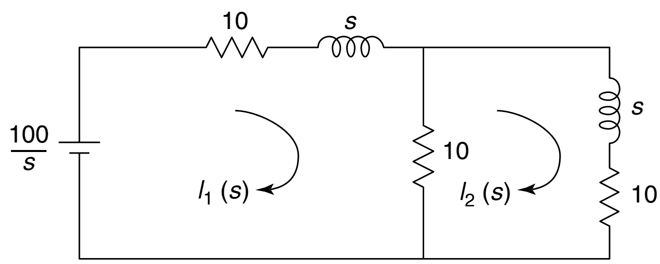



**Fig. 11.58**

Applying KVL to Mesh 1,

$$\frac{100}{s} - 10I_1(s) - sI_1(s) - 10\left[I_1(s) - I_2(s)\right] = 0$$

$$(s+20)I_1(s) - 10I_2(s) = \frac{100}{s}$$

Applying KVL to Mesh 2,

$$-10\left[I_2(s) - I_1(s)\right] - s\,I_2(s) - 10\,I_2(s) = 0$$

$$-10\,I_1(s) + (s+20)I_2(s) = 0$$

By Cramer's rule,

$$I_1(s) = \frac{\begin{vmatrix} \frac{100}{s} & -10 \\ 0 & s+20 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{\frac{100}{s}(s+20)}{(s+20)^2 - 100} = \frac{100(s+20)}{s(s^2+40s+300)} = \frac{100(s+20)}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_1(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{s+30}$$

$$A = s\, I_1(s)\big|_{s=0} = \left.\frac{100(s+20)}{(s+10)(s+30)}\right|_{s=0} = \frac{20}{3}$$

$$B = (s+10)I_1(s)\big|_{s=-10} = \left.\frac{100(s+20)}{s(s+30)}\right|_{s=-10} = -5$$

$$C = (s+30)I_1(s)\big|_{s=-30} = \left.\frac{100(s+20)}{s(s+10)}\right|_{s=-30} = -\frac{5}{3}$$

$$I_1(s) = \frac{20}{3}\frac{1}{s} - \frac{5}{s+10} - \frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_1(t) = \frac{20}{3} - 5e^{-10t} - \frac{5}{3}e^{-30t}$$

Similarly,

$$I_2(s) = \frac{\begin{vmatrix} s+20 & \frac{100}{s} \\ -10 & 0 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{\frac{1000}{s}}{(s+20)^2 - 100} = \frac{1000}{s(s^2+40s+300)} = \frac{1000}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_2(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{s+30}$$

$$A = sI_2(s)\big|_{s=0} = \left.\frac{1000}{(s+10)(s+30)}\right|_{s=0} = \frac{10}{3}$$

$$B = (s+10)I_2(s)\big|_{s=-10} = \left.\frac{1000}{s(s+30)}\right|_{s=-10} = -5$$

$$C = (s+30)I_2(s)\big|_{s=-30} = \left.\frac{1000}{s(s+10)}\right|_{s=-30} = \frac{5}{3}$$

$$I_2(s) = \frac{10}{3}\frac{1}{s} - \frac{5}{s+10} + \frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_2(t) = \frac{10}{3} - 5e^{-10t} + \frac{5}{3}e^{-30t} \qquad \text{for } t > 0$$

## 11.12 RESISTOR–CAPACITOR CIRCUIT

Consider a series *RC* circuit as shown in Fig. 11.59. The switch is closed at time $t = 0$.

For $t > 0$, the transformed network is shown in Fig. 11.60.

Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - \frac{1}{Cs}I(s) = 0$$

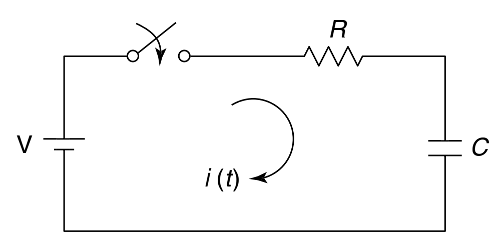



**Fig. 11.59** *RC circuit*

$$\left(R+\frac{1}{Cs}\right)I(s)=\frac{V}{s}$$

$$I(s)=\frac{\frac{V}{s}}{R+\frac{1}{Cs}}=\frac{\frac{V}{s}}{\frac{RCs+1}{Cs}}=\frac{\frac{V}{R}}{s+\frac{1}{RC}}$$

Taking the inverse Laplace transform,

$$i(t)=\frac{V}{R}e^{-\frac{1}{RC}t} \qquad \text{for } t>0$$

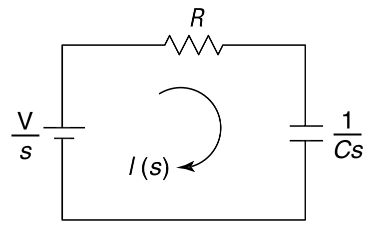



**Fig. 11.60** *Transformed network*

**Example 11.86** *In the network of Fig. 11.61, the switch is moved from a to b at t = 0. Determine i (t) and $v_c$ (t).*

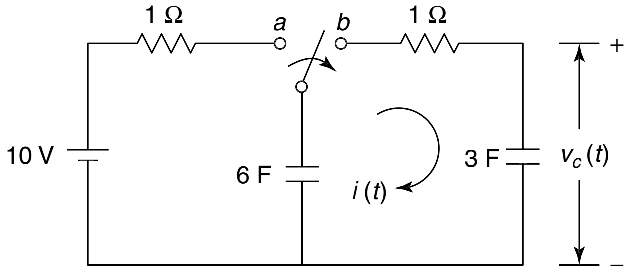



**Fig. 11.61**

**Solution** At $t=0^-$, the network is shown in Fig. 11.62. At $t=0^-$, the network has attained steady-state condition. Hence, the capacitor of 6 F acts as an open circuit.

$$v_{6\,\text{F}}(0^-)=10\text{ V}$$
$$i(0^-)=0$$
$$v_{3\,\text{F}}(0^-)=0$$

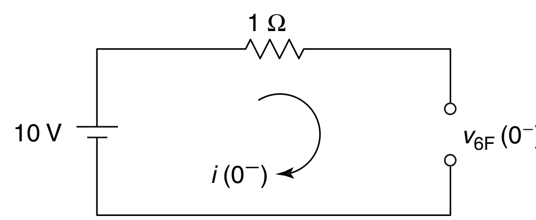



**Fig. 11.62**

Since voltage across the capacitor cannot change instantaneously,

$$v_{6\,\text{F}}(0^+)=10\text{ V}$$
$$v_{3\,\text{F}}(0^+)=0$$

For $t>0$, the transformed network is shown in 11.63.

Applying KVL to the mesh for $t>0$,

$$\frac{10}{s}-\frac{1}{6s}I(s)-I(s)-\frac{1}{3s}I(s)=0$$

$$\frac{1}{6s}I(s)+I(s)+\frac{1}{3s}I(s)=\frac{10}{s}$$

$$I(s)=\frac{10}{s\left(1+\frac{1}{6s}+\frac{1}{3s}\right)}=\frac{60}{6s+3}=\frac{10}{s+0.5}$$

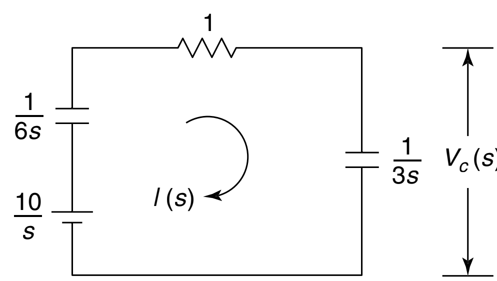



**Fig. 11.63**

Taking the inverse Laplace transform,

$$i(t)=10e^{-0.5t} \qquad \text{for } t>0$$

Voltage across the 3 F capacitor is given by

$$V_c(s)=\frac{1}{3s}I(s)=\frac{10}{3s(s+0.5)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.5}$$

$$A = sV_c(s)\big|_{s=0} = \left.\frac{10}{3(s+0.5)}\right|_{s=0} = \frac{20}{3}$$

$$B = (s+0.5)V_c(s)\big|_{s=-0.5} = \left.\frac{10}{3s}\right|_{s=-0.5} = -\frac{20}{3}$$

$$V_c(s) = \frac{20}{3}\frac{1}{s} - \frac{20}{3}\frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v_c(t) = \frac{20}{3} - \frac{20}{3}e^{-0.5t}$$

$$= \frac{20}{3}(1-e^{-0.5t}) \qquad \text{for } t > 0$$

**Example 11.87** *The switch in the network shown in Fig. 11.64 is closed at t = 0. Determine the voltage cross the capacitor.*

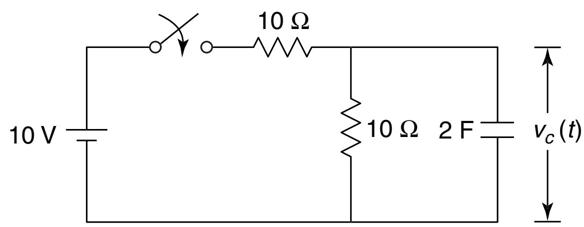



**Fig. 11.64**

**Solution** At $t = 0^-$, the capacitor is uncharged.

$$v_c(0^-) = 0$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 11.65.

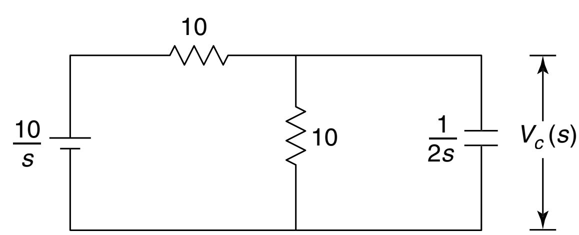



**Fig. 11.65**

Applying KCL at the node for $t > 0$,

$$\frac{V_c(s) - \frac{10}{s}}{10} + \frac{V_c(s)}{10} + \frac{V_c(s)}{\frac{1}{2s}} = 0$$

$$2sV_c(s) + 0.2V_c(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s(2s+0.2)} = \frac{0.5}{s(s+0.1)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.1}$$

$$A = sV_c(s)\big|_{s=0} = \left.\frac{0.5}{s+0.1}\right|_{s=0} = \frac{0.5}{0.1} = 5$$

$$B = (s+0.1)V_c(s)\Big|_{s=-0.1} = \frac{0.5}{s}\Big|_{s=-0.1} = -\frac{0.5}{0.1} = -5$$

$$V_c(s) = \frac{5}{s} - \frac{5}{s+0.1}$$

Taking inverse Laplace transform,

$$v_c(t) = 5 - 5e^{-0.1t} \qquad \text{for } t > 0$$

**Example 11.88** *In the network of Fig. 11.66, the switch is closed for a long time and at t = 0, the switch is opened. Determine the current through the capacitor.*

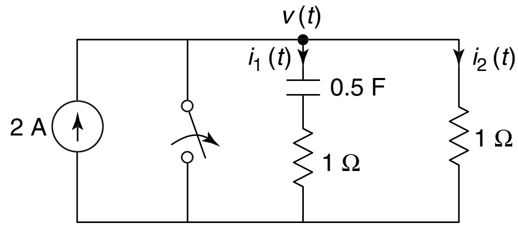



**Fig. 11.66**

**Solution** At $t = 0^-$, the network is shown in Fig. 11.67. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the capacitor acts as an open circuit.

$$v_c(0^-) = 0$$

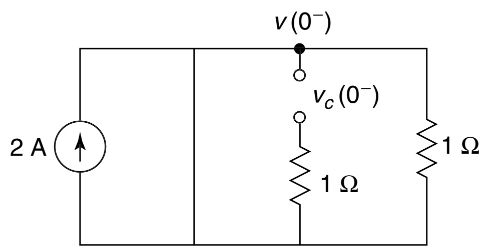



**Fig. 11.67**

Since voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 11.68.

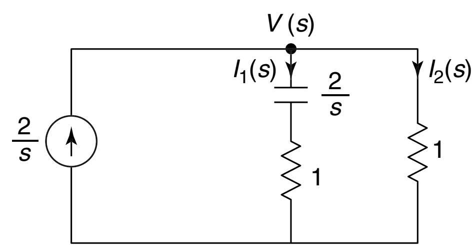



**Fig. 11.68**

Applying KVL to two parallel branches,

$$\frac{2}{s}I_1(s) + I_1(s) = I_2(s)$$

Applying KCL at the node for $t > 0$,

$$\frac{2}{s} = I_1(s) + I_2(s)$$

$$\frac{2}{s}I_1(s) + I_1(s) = \frac{2}{s} - I_1(s)$$

$$\frac{2}{s}I_1(s) + 2I_1(s) = \frac{2}{s}$$

$$I_1(s) = \frac{\frac{2}{s}}{\frac{2}{s}+2} = \frac{1}{s+1}$$

Taking the inverse Laplace transform,

$$i_1(t) = e^{-t} \qquad \text{for } t > 0$$

**Example 11.89** *In the network of Fig. 11.69, the switch is moved from a to b, at t = 0. Find v(t).*

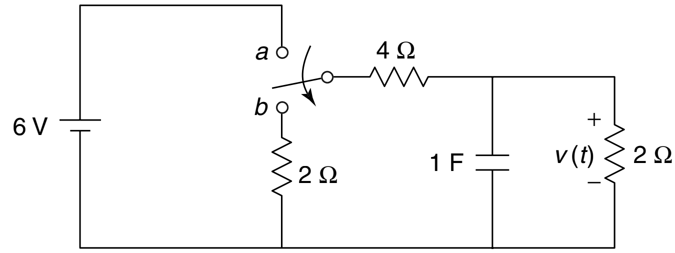



**Fig. 11.69**

**Solution** At $t = 0^-$, the network is shown in Fig 11.70. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor acts as an open circuit.

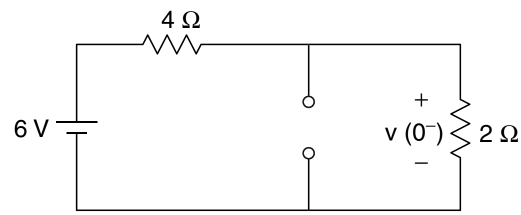



**Fig. 11.70**

$$v(0^-) = 6 \times \frac{2}{4+2} = 2 \text{ V}$$

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = 2 \text{ V}$$

For $t > 0$, the transformed network is shown in Fig. 11.71.

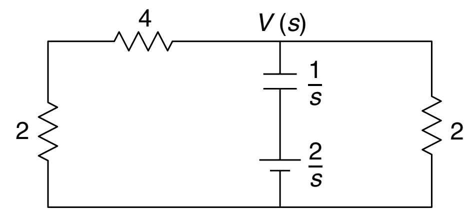



**Fig. 11.71**

Applying KCL at the node for $t > 0$,

$$\frac{V(s)}{6} + \frac{V(s) - \frac{2}{s}}{\frac{1}{s}} + \frac{V(s)}{2} = 0$$

$$V(s)\left(\frac{2}{3} + s\right) = 2$$

$$V(s) = \frac{2}{s + \frac{2}{3}}$$

Taking the inverse Laplace transform,

$$v(t) = 2e^{-\frac{2}{3}t} \qquad \text{for } t > 0$$

**Example 11.90** *The network shown in Fig. 11.72 has acquired steady-state at t < 0 with the switch open. The switch is closed at t = 0. Determine v (t).*

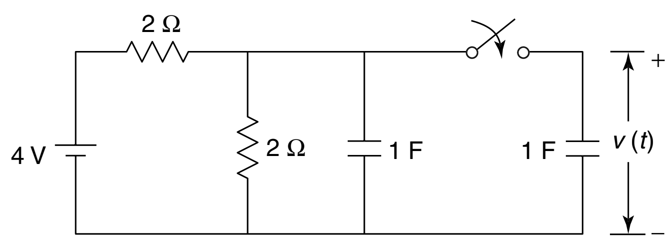



**Fig. 11.72**

**Solution** At $t = 0^-$, the network is shown in Fig 11.73. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor of 1 F acts as an open circuit.

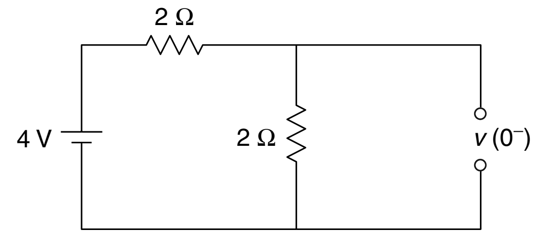



**Fig. 11.73**

$$v(0^-) = 4 \times \frac{2}{2+2} = 2 \text{ V}$$

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = 2 \text{ V}$$

For $t > 0$, the transformed network is shown in Fig. 11.74.

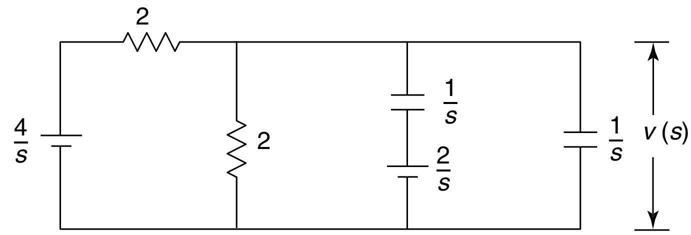



**Fig. 11.74**

Applying KCL at the node for $t > 0$,

$$\frac{V(s)-\frac{4}{s}}{2} + \frac{V(s)}{2} + \frac{V(s)-\frac{2}{s}}{\frac{1}{s}} + \frac{V(s)}{\frac{1}{s}} = 0$$

$$2sV(s) + V(s) = \frac{2}{s} + 2$$

$$V(s) = \frac{\frac{2}{s}+2}{2s+1} = \frac{2s+2}{s(2s+1)} = \frac{2}{s} - \frac{2}{2s+1} = \frac{2}{s} - \frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v(t) = 2 - e^{-0.5t} \qquad \text{for } t > 0$$

## 11.13 RESISTOR–INDUCTOR–CAPACITOR CIRCUIT

Consider a series *RLC* circuit shown in Fig. 11.75. The switch is closed at time $t = 0$.

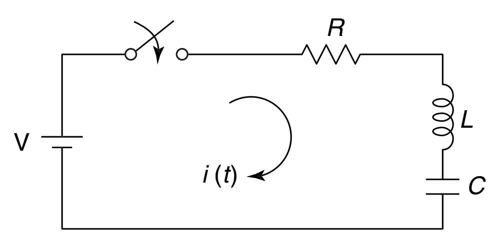



**Fig. 11.75** *RLC circuit*

For $t > 0$, the transformed network is shown in Fig. 11.76.

Applying KVL to the mesh,

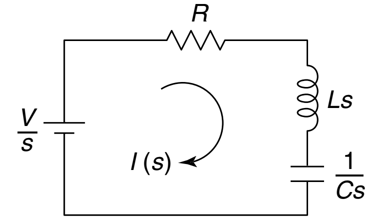



**Fig. 11.76** *Transformed network*

$$\frac{V}{s} - RI(s) - Ls\,I(s) - \frac{1}{Cs}I(s) = 0$$

$$\left(R + Ls + \frac{1}{Cs}\right)I(s) = \frac{V}{s}$$

$$\left(\frac{LCs^2 + RCs + 1}{Cs}\right)I(s) = \frac{V}{s}$$

$$I(s) = \frac{\frac{V}{s}}{\frac{LCs^2 + RCs + 1}{Cs}} = \frac{\frac{V}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\frac{V}{L}}{(s - s_1)(s - s_2)}$$

where $s_1$ and $s_2$ are the roots of the equation $s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right) = 0$.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \beta$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \beta$$

where $$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and $$\beta = \sqrt{\alpha^2 - \omega_0^2}$$

By partial-fraction expansion, of $I(s)$,

$$I(s) = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

$$A = (s - s_1)\,I(s)\Big|_{s = s_1} = \frac{\frac{V}{L}}{s_1 - s_2}$$

$$B = (s - s_2)\,I(s)\Big|_{s = s_2} = \frac{\frac{V}{L}}{s_2 - s_1} = -\frac{\frac{V}{L}}{s_1 - s_2}$$

$$I(s) = \frac{V}{L(s_1 - s_2)}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right]$$

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{L(s_1 - s_2)}\left[e^{s_1 t} - e^{s_2 t}\right] = k_1\,e^{s_1 t} + k_2\,e^{s_2 t}$$

where $k_1$ and $k_2$ are constants to be determined and $s_1$ and $s_2$ are the roots of the equation.

Now depending upon the values of $s_1$ and $s_2$, we have 3 cases of the response.

***Case I*** When the roots are real and unequal, it gives an overdamped response.

$$\frac{R}{2L} > \frac{1}{\sqrt{LC}}$$

$$\alpha > \omega_0$$

In this case, the solution is given by

$$i(t) = e^{-\alpha t}\left(k_1 e^{\beta t} + k_2 e^{-\beta t}\right)$$

or $$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} \qquad \text{for } t > 0$$

***Case II*** When the roots are real and equal, it gives a critically damped response.

$$\frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

$$\alpha = \omega_0$$

In this case, the solution is given by

$$i(t) = e^{-\alpha t}(k_1 + k_2 t) \qquad \text{for } t > 0$$

***Case III*** When the roots are complex conjugate, it gives an underdamped response.

$$\frac{R}{2L} < \frac{1}{\sqrt{LC}}$$

$$\alpha < \omega_0$$

In this case, the solution is given by

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Let $$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\omega_d$$

where $$j = \sqrt{-1}$$

and $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Hence

$$i(t) = e^{-\alpha t}\left(k_1 e^{j\omega_d t} + k_2 e^{-j\omega_d t}\right)$$

$$= e^{-\alpha t}\left[(k_1 + k_2)\left\{\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right\} + j(k_1 - k_2)\left\{\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}\right\}\right]$$

$$= e^{-\alpha t}\left[(k_1 + k_2)\cos\omega_d t + j(k_1 - k_2)\sin\omega_d t\right] \qquad \text{for } t > 0$$

**Example 11.91** *The switch in Fig. 11.77 is opened at time $t = 0$. Determine the voltage $v(t)$ for $t > 0$.*

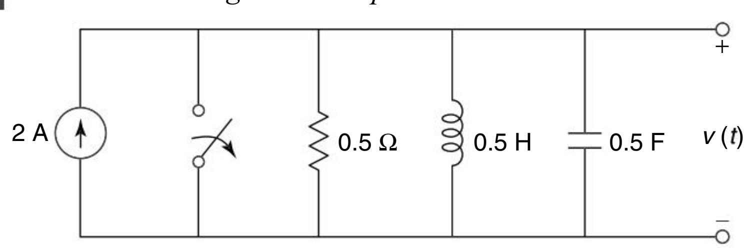



**Fig. 11.77**

**Solution** At $t = 0^-$, the network is shown in Fig. 11.78. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

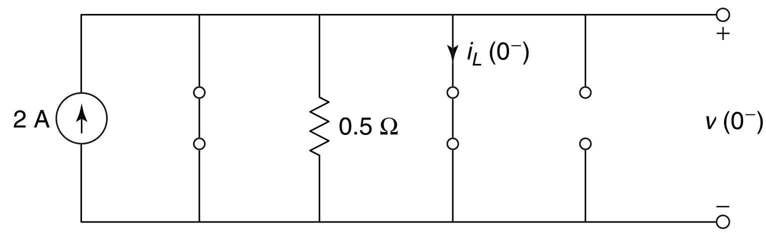



**Fig. 11.78**

$$i_L(0^-) = 0$$
$$v(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$i_L(0^+) = 0$$
$$v(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 11.79.

Applying KCL at the node for $t > 0$,

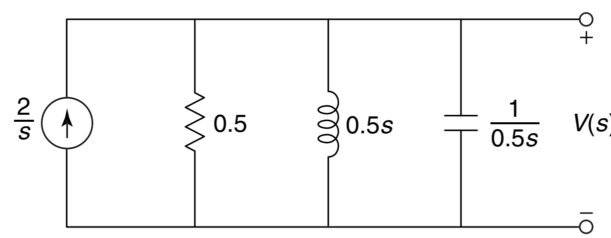



**Fig. 11.79**

$$\frac{V(s)}{0.5} + \frac{V(s)}{0.5s} + \frac{V(s)}{\frac{1}{0.5s}} = \frac{2}{s}$$

$$2V(s) + \frac{2}{s}V(s) + 0.5sV(s) = \frac{2}{s}$$

$$V(s) = \frac{\frac{2}{s}}{\frac{2}{s} + 0.5s + 2} = \frac{4}{s^2 + 4s + 4} = \frac{4}{(s+2)^2}$$

Taking inverse Laplace transform,

$$v(t) = 4t\, e^{-2t} \qquad \text{for } t > 0$$

**Example 11.92** *In the network of Fig. 11.80, the switch is closed and steady-state is attained. At t = 0, switch is opened. Determine the current through the inductor.*

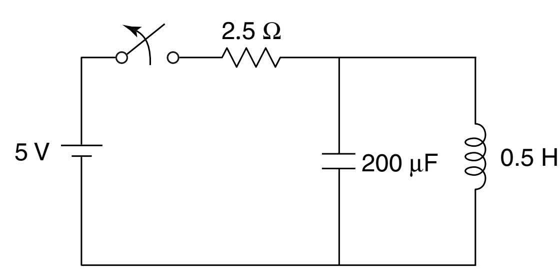



**Fig. 11.80**

**Solution** At $t = 0^-$, the network is shown in Fig. 11.81. At $t = 0^-$, the switch is closed and steady-state condition is attained. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit. Current through inductor is same as the current through the resistor.

$$i_L(0^-) = \frac{5}{2.5} = 2 \text{ A}$$

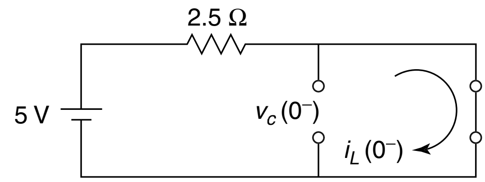



**Fig. 11.81**

Voltage across the capacitor is zero as it is connected in parallel with a short.

$$v_c(0^-) = 0$$

Since voltage across the capacitor and current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2 \text{ A}$$
$$v_c(0^+) = 0$$

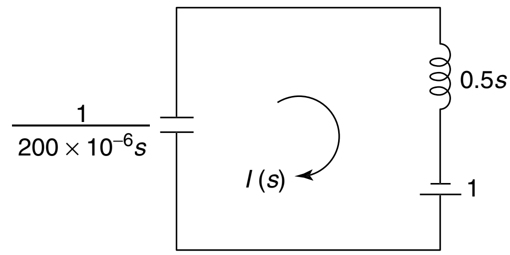



**Fig. 11.82**

For $t > 0$, the transformed network is shown in Fig. 11.82. Applying KVL to the mesh for $t > 0$,

$$-\frac{1}{200 \times 10^{-6} s} I(s) - 0.5s\, I(s) + 1 = 0$$

$$0.5s\, I(s) - 1 + 5000 \frac{I(s)}{s} = 0$$

$$I(s) = \frac{1}{0.5s + \frac{5000}{s}} = \frac{2s}{s^2 + 10000}$$

Taking inverse Laplace transform,

$$i(t) = 2 \cos 100t \qquad \text{for } t > 0$$

**Example 11.93** *In the network shown in Fig. 11.83, the switch is opened at t = 0. Steady-state condition is achieved before t = 0. Find i(t).*

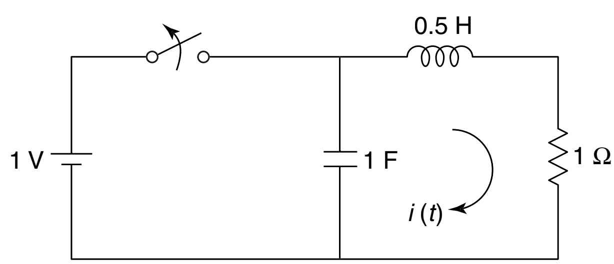



**Fig. 11.83**

***Solution*** At $t = 0^-$, the network is shown in Fig 11.84. At $t = 0^-$, the switch is closed and steady-state condition is achieved. Hence, the capacitor acts as an open circuit and the inductor acts as a short circuit.

$$v_c(0^-) = 1 \text{ V}$$
$$i(0^-) = 1 \text{ A}$$

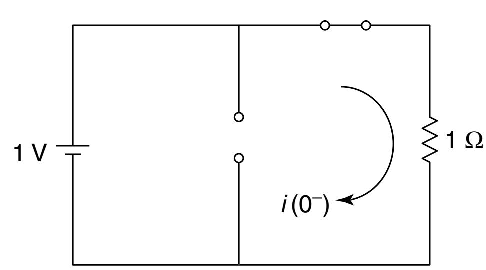



**Fig. 11.84**

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 1 \text{ V}$$
$$i(0^+) = 1 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 11.85.

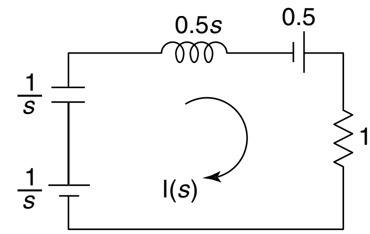



**Fig. 11.85**

Applying KVL to the mesh for $t > 0$,

$$\frac{1}{s} - \frac{1}{s} I(s) - 0.5s\, I(s) + 0.5 - I(s) = 0$$

$$0.5 + \frac{1}{s} = \frac{1}{s} I(s) + 0.5\, s\, I(s) + I(s)$$

$$I(s)\left[1 + \frac{1}{s} + 0.5s\right] = 0.5 + \frac{1}{s}$$

$$I(s) = \frac{s+2}{s^2+2s+2} = \frac{(s+1)+1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

Taking the inverse Laplace transform,

$$i(t) = e^{-t}\cos t + e^{-t}\sin t \qquad \text{for } t > 0$$

**Example 11.94** *In the network shown in Fig. 11.86, the switch is closed at t = 0. Find the currents $i_1(t)$ and $i_2(t)$ when initial current through the inductor is zero and initial voltage on the capacitor is 4 V.*

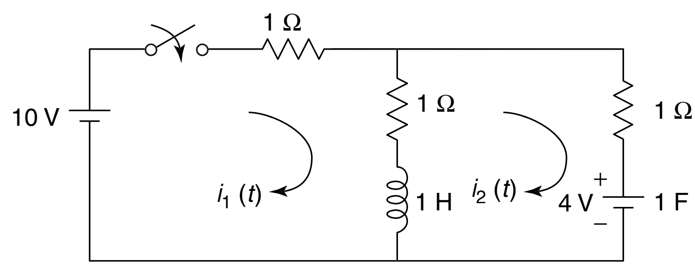



**Fig. 11.86**

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.87.

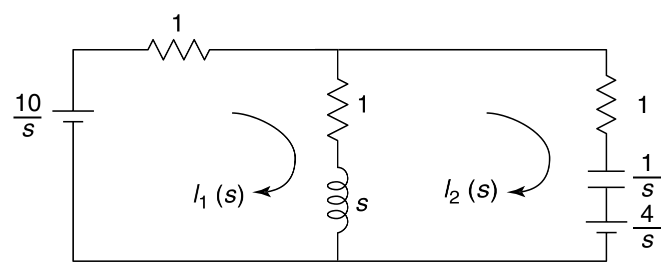



**Fig. 11.87**

Applying KVL to Mesh 1,

$$\frac{10}{s} - I_1(s) - (1+s)\left[I_1(s) - I_2(s)\right] = 0$$

$$(s+2)I_1(s) - (s+1)I_2(s) = \frac{10}{s}$$

Applying KVL to Mesh 2,

$$-(s+1)\left[I_2(s)-I_1(s)\right]-I_2(s)-\frac{1}{s}I_2(s)-\frac{4}{s}=0$$

$$-(s+1)I_1(s)+\left(s+2+\frac{1}{s}\right)I_2(s)=-\frac{4}{s}$$

By Cramer's rule,

$$I_1(s)=\frac{\begin{vmatrix}\frac{10}{s} & -(s+1)\\ -\frac{4}{s} & s+2+\frac{1}{s}\end{vmatrix}}{\begin{vmatrix}s+2 & -(s+1)\\ -(s+1) & s+2+\frac{1}{s}\end{vmatrix}}=\frac{\left(\frac{10}{s}\right)\left(\frac{s^2+2s+1}{s}\right)-(s+1)\left(\frac{4}{s}\right)}{(s+2)\left(\frac{s^2+2s+1}{s}\right)-(s+1)^2}=\frac{\frac{10}{s^2}(s+1)^2-(s+1)\frac{4}{s}}{(s+2)\frac{(s+1)^2}{s}-(s+1)^2}$$

$$=\frac{\frac{10}{s^2}(s+1)-\frac{4}{s}}{(s+2)\frac{(s+1)}{s}-(s+1)}=\frac{3s+5}{s(s+1)}$$

By partial-fraction expansion,

$$I_1(s)=\frac{A}{s}+\frac{B}{s+1}$$

$$A=sI_1(s)\big|_{s=0}=\left.\frac{3s+5}{s+1}\right|_{s=0}=5$$

$$B=(s+1)I_1(s)\big|_{s=-1}=\left.\frac{3s+5}{s}\right|_{s=-1}=-2$$

$$I_1(s)=\frac{5}{s}-\frac{2}{s+1}$$

Taking inverse Laplace transform,

$$i_1(t)=5-2e^{-t} \qquad \text{for } t>0$$

Similarly,

$$I_2(s)=\frac{\begin{vmatrix}s+2 & \frac{10}{s}\\ -(s+1) & -\frac{4}{s}\end{vmatrix}}{\begin{vmatrix}s+2 & -(s+1)\\ -(s+1) & s+2+\frac{1}{s}\end{vmatrix}}=\frac{3s+1}{(s+1)^2}=\frac{3s+3-2}{(s+1)^2}=\frac{3(s+1)-2}{(s+1)^2}=\frac{3}{s+1}-\frac{2}{(s+1)^2}$$

Taking inverse Laplace transform,

$$i_2(t)=3e^{-t}-2te^{-t} \qquad \text{for } t>0$$

## 11.14 RESPONSE OF RL CIRCUIT TO VARIOUS FUNCTIONS

Consider a series *RL* circuit shown in Fig. 11.88. When the switch is closed at $t=0$, $i(0^-)=i(0^+)=0$.

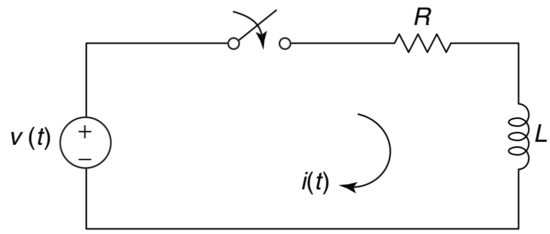



**Fig. 11.88** *RL circuit*

For $t > 0$, the transformed network is shown in Fig. 11.89.

Applying KVL to the mesh,

$$V(s) - R\,I(s) - Ls\,I(s) = 0$$

$$I(s) = \frac{V(s)}{R + Ls} = \frac{1}{L}\frac{V(s)}{s + \frac{R}{L}}$$

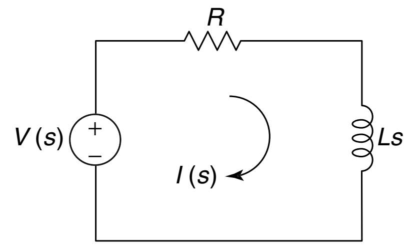



**Fig. 11.89** *Transformed network*

(a) When the unit step signal is applied,

$$v(t) = u(t)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}$$

$$I(s) = \frac{1}{L}\frac{\frac{1}{s}}{s + \frac{R}{L}}$$

$$= \frac{1}{L}\frac{1}{s\left(s + \frac{R}{L}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{1}{L}\left(\frac{A}{s} + \frac{B}{s + \frac{R}{L}}\right)$$

$$A = s\,I(s)\Big|_{s=0} = \frac{1}{s + \frac{R}{L}}\Bigg|_{s=0} = \frac{L}{R}$$

$$B = \left(s + \frac{R}{L}\right)I(s)\Big|_{s=-\frac{R}{L}} = \frac{1}{s}\Bigg|_{s=-\frac{R}{L}} = -\frac{L}{R}$$

$$I(s) = \frac{1}{L}\left(\frac{L}{R}\frac{1}{s} - \frac{L}{R}\frac{1}{s + \frac{R}{L}}\right)$$

$$= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s + \frac{R}{L}}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}[1 - e^{-\left(\frac{R}{L}\right)t}] \qquad \text{for } t > 0$$

(b) When unit ramp signal is applied,

$$v(t) = r(t) = t \qquad \text{for } t > 0$$

Taking Laplace transform,

$$V(s) = \frac{1}{s^2}$$

$$I(s) = \frac{1}{L}\frac{1}{s^2\left(s + \frac{R}{L}\right)}$$

By partial-faction expansion,

$$\frac{1}{L}\frac{1}{s^2\left(s + \frac{R}{L}\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + \frac{R}{L}}$$

$$\frac{1}{L} = As\left(s + \frac{R}{L}\right) + B\left(s + \frac{R}{L}\right) + Cs^2$$

Putting $s = 0$,

$$B = \frac{1}{R}$$

Putting $s = -\frac{R}{L}$,

$$C = \frac{L}{R^2}$$

Comparing coefficients of $s^2$,

$$A + C = 0$$

$$A = -C = -\frac{L}{R^2}$$

$$I(s) = -\frac{L}{R^2}\frac{1}{s} + \frac{1}{R}\frac{1}{s^2} + \frac{L}{R^2}\frac{1}{s + \frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = -\frac{L}{R^2} + \frac{1}{R}t + \frac{L}{R^2}e^{-\left(\frac{R}{L}\right)t}$$

$$= \frac{1}{R}t - \frac{L}{R^2}[1 - e^{-\left(\frac{R}{L}\right)t}] \qquad \text{for } t > 0$$

(c) When unit impulse signal is applied,

$$v(t) = \delta(t)$$

Taking Laplace transform,

$$V(s) = 1$$

$$I(s) = \frac{1}{L}\frac{1}{s + \frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{L} e^{-\left(\frac{R}{L}\right)t} \qquad \text{for } t > 0$$

**Example 11.95** *At t = 0, unit pulse voltage of unit width is applied to a series RL circuit as shown in Fig. 11.90. Obtain an expression for i(t).*

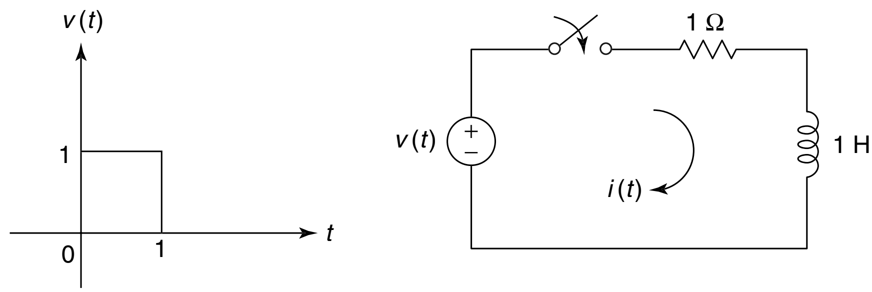



**Fig. 11.90**

**Solution**

$$v(t) = u(t) - u(t-1)$$

$$V(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s}$$

For $t > 0$, the transformed network is shown in Fig. 11.91.

Applying KVL to the mesh,

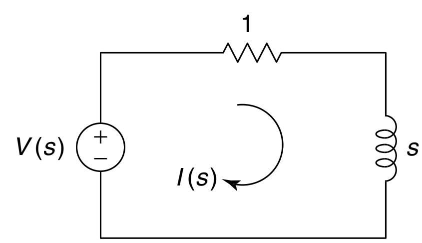



**Fig. 11.91**

$$V(s) - I(s) - sI(s) = 0$$

$$\begin{aligned} I(s) &= \frac{V(s)}{s+1} \\ &= \frac{1-e^{-s}}{s(s+1)} \\ &= \frac{1}{s(s+1)} - \frac{e^{-s}}{s(s+1)} \\ &= \frac{1}{s} - \frac{1}{s+1} - \frac{e^{-s}}{s} + \frac{e^{-s}}{s+1} \end{aligned}$$

Taking inverse Laplace transform,

$$\begin{aligned} i(t) &= u(t) - e^{-t}u(t) - u(t-1) + e^{-(t-1)}u(t-1) \\ &= (1-e^{-t})u(t) - [1-e^{-(t-1)}]u(t-1) \qquad \text{for } t > 0 \end{aligned}$$

**Example 11.96** *For the network shown in Fig. 11.92, determine the current i(t) when the switch is closed at t = 0. Assume that initial current in the inductor is zero.*

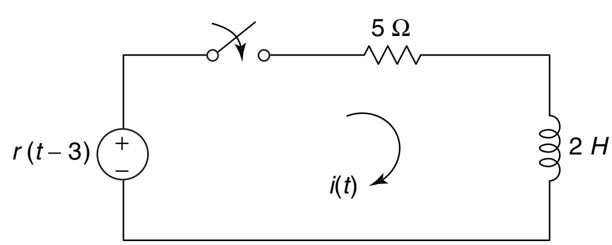


Fig. 11.92

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.93. Applying KVL to the mesh for $t > 0$,

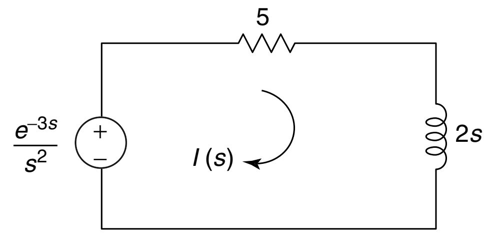



Fig. 11.93

$$\frac{e^{-3s}}{s^2} - 5\,I(s) - 2s\,I(s) = 0$$

$$5\,I(s) + 2s\,I(s) = \frac{e^{-3s}}{s^2}$$

$$I(s) = \frac{e^{-3s}}{s^2(2s+5)} = \frac{0.5\,e^{-3s}}{s^2(s+2.5)}$$

By partial-fraction expansion,

$$\frac{0.5}{s^2(s+2.5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2.5}$$

$$0.5 = As(s+2.5) + B(s+2.5) + Cs^2$$

$$= As^2 + 2.5As + Bs + 2.5B + Cs^2$$

$$= (A+C)s^2 + (2.5A+B)s + 2.5\,B$$

Comparing coefficients of $s^2$, $s$ and $s^0$,

$$A + C = 0$$

$$2.5A + B = 0$$

$$2.5B = 0.5$$

Solving these equations,

$$A = -0.08$$

$$B = 0.2$$

$$C = 0.08$$

$$I(s) = e^{-3s}\left(-\frac{0.08}{s} + \frac{0.2}{s^2} + \frac{0.08}{s+2.5}\right)$$

$$= -0.08\frac{e^{-3s}}{s} + 0.2\frac{e^{-3s}}{s^2} + 0.08\frac{e^{-3s}}{s+2.5}$$

Taking inverse Laplace transform,

$$i(t) = -0.08u(t-3) + 0.2r(t-3) + 0.08e^{-2.5(t-3)}u(t-3)$$

**Example 11.97** *Determine the expression for $v_L(t)$ in the network shown in Fig. 11.94. Find $v_L(t)$ when (i) $v_s(t) = \delta(t)$, and (ii) $v_s(t) = e^{-t}u(t)$.*

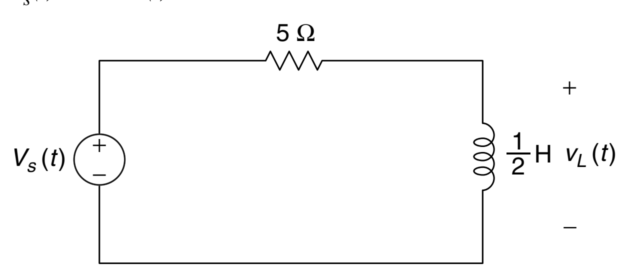



Fig. 11.94

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.95.
By voltage-division rule,

$$V_L(s) = V_s(s) \times \frac{\frac{s}{2}}{\frac{s}{2}+5} = \frac{s}{s+10} V_s(s)$$

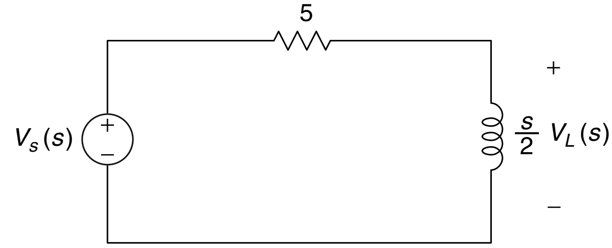



**Fig. 11.95**

(a) For impulse input,

$$V_s(s) = 1$$

$$V_L(s) = \frac{s}{s+10} = \frac{s+10-10}{s+10} = 1 - \frac{10}{s+10}$$

Taking inverse Laplace transform,

$$V_L(t) = \delta(t) - 10e^{-10t}u(t) \qquad \text{for } t > 0$$

(b) For $v_s(t) = e^{-t}u(t)$,

$$V_s(s) = \frac{1}{s+1}$$

$$V_L(s) = \frac{s}{(s+10)(s+1)}$$

By partial-fraction expansion,

$$V_L(s) = \frac{A}{s+10} + \frac{B}{s+1}$$

$$A = (s+10)V_L(s)\Big|_{s=-10} = \frac{s}{s+1}\Big|_{s=-10} = \frac{10}{9}$$

$$B = (s+1)V_L(s)\Big|_{s=-1} = \frac{s}{s+10}\Big|_{s=-1} = -\frac{1}{9}$$

$$V_L(s) = \frac{10}{9}\frac{1}{s+10} - \frac{1}{9}\frac{1}{s+1}$$

Taking inverse Laplace transform,

$$v_L(t) = \frac{10}{9}e^{-10t}u(t) - \frac{1}{9}e^{-t}u(t)$$

$$= \left(\frac{10}{9}e^{-10t} - \frac{1}{9}e^{-t}\right)u(t) \qquad \text{for } t > 0$$

**Example 11.98** *For the network shown in Fig. 11.96, determine the current i (t) when the switch is closed at t = 0. Assume that initial current in the inductor is zero.*

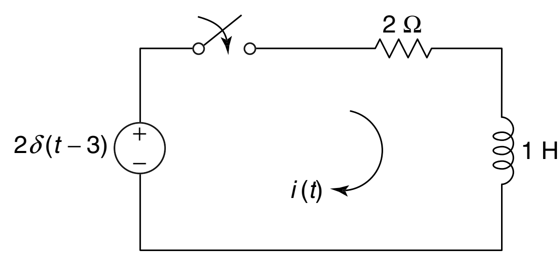



**Fig. 11.96**

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.97.

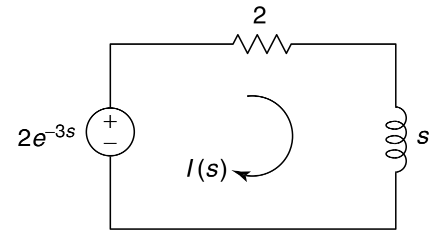



**Fig. 11.97**

Applying KVL to the mesh for $t > 0$,

$$2e^{-3s} - 2I(s) - sI(s) = 0$$

$$2I(s) + sI(s) = 2e^{-3s}$$

$$I(s) = \frac{2e^{-3s}}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = 2e^{-2(t-3)}u(t-3) \qquad \text{for } t > 0$$

**Example 11.99** *Determine the current i(t) in the network shown in Fig. 11.98, when the switch is closed at t = 0.*

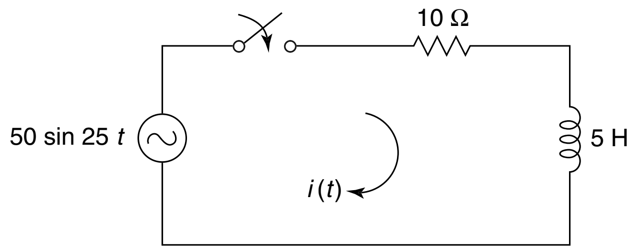



**Fig. 11.98**

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.99.

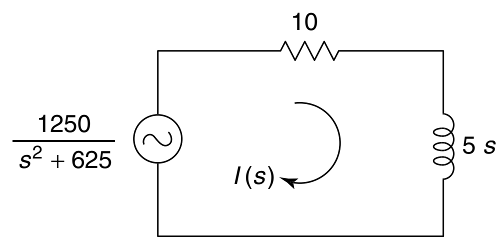



**Fig. 11.99**

Applying KVL to the mesh for $t > 0$,

$$\frac{1250}{s^2 + 625} - 10I(s) - 5sI(s) = 0$$

$$I(s) = \frac{250}{(s^2 + 625)(s+2)}$$

By partial-fraction expansion,

$$I(s) = \frac{As + B}{s^2 + 625} + \frac{C}{s+2}$$

$$250 = (As + B)(s+2) + C(s^2 + 625)$$

$$= (A + C)s^2 + (2A + B)s + (2B + 625C)$$

Comparing coefficients,

$$A + C = 0$$

$$2A + B = 0$$

$$2B + 625C = 250$$

Solving the equations,

$$A = -0.397$$

$$B = 0.795$$

$$C = 0.397$$

$$I(s) = \frac{-0.397s + 0.795}{s^2 + 625} + \frac{0.397}{s+2} = -\frac{0.397s}{s^2 + 625} + \frac{0.795}{s^2 + 625} + \frac{0.397}{s+2}$$

Taking the inverse Laplace transform,

$$i(t) = -0.397\cos 25t + 0.032\sin 25t + 0.397e^{-2t} \qquad \text{for } t > 0$$

**Example 11.100** *Find impulse response of the current i(t) in the network shown in Fig. 11.100.*

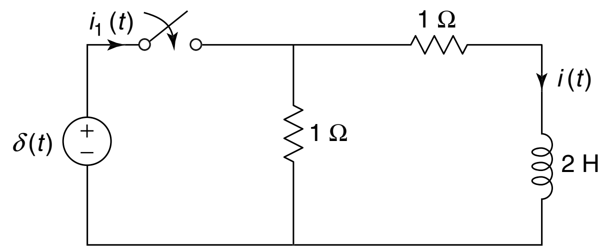



**Fig. 11.100**

**Solution** The transformed network is shown in Fig. 11.101.

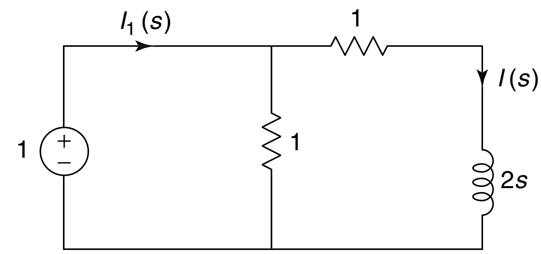



**Fig. 11.101**

$$Z(s) = \frac{1(2s+1)}{2s+1+1} = \frac{2s+1}{2s+2}$$

$$I_1(s) = \frac{V(s)}{Z(s)} = \frac{1}{\dfrac{2s+1}{2s+2}} = \frac{2s+2}{2s+1}$$

By current-division rule,

$$I(s) = I_1(s) \times \frac{1}{2s+2} = \frac{1}{2s+2} \times \frac{2s+2}{2s+1} = \frac{1}{2s+1} = \frac{1}{2}\frac{1}{s+0.5}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{2}e^{-0.5t}u(t) \qquad \text{for } t > 0$$

**Example 11.101** *The network shown in Fig. 11.102 is at rest for $t < 0$. If the voltage $v(t) = u(t)\cos t + A\delta(t)$ is applied to the network, determine the value of A so that there is no transient term in the current response i(t).*

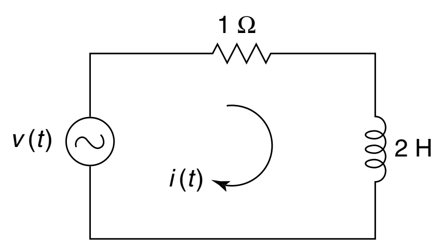



**Fig. 11.102**

$$v(t) = u(t)\cos t + A\delta(t)$$

$$V(s) = \frac{s}{s^2+1} + A$$

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.103.
Applying KVL to the mesh for $t > 0$,

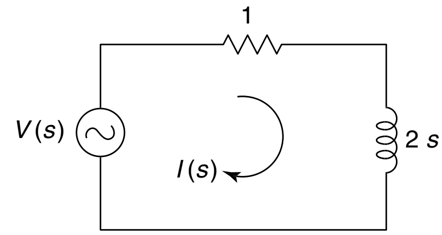



**Fig. 11.103**

$$V(s) = 2sI(s) + I(s) = \frac{s}{s^2+1} + A$$

$$I(s) = \frac{s + A(s^2+1)}{2\left(s+\frac{1}{2}\right)(s^2+1)} = \frac{K_1}{s+\frac{1}{2}} + \frac{K_2 s + K_3}{s^2+1}$$

The transient part of the response is given by the first term. Hence, for the transient term to vanish, $K_1 = 0$.

$$K_1 = \left(s+\frac{1}{2}\right)I(s)\bigg|_{s=-\frac{1}{2}} = \frac{\frac{-1}{2} + A\left(\frac{5}{4}\right)}{2\left(\frac{5}{4}\right)}$$

When $K_1 = 0$

$$\frac{5}{4}A = \frac{1}{2}$$

$$A = \frac{2}{5} = 0.4$$

## 11.15 RESPONSE OF RC CIRCUIT TO VARIOUS FUNCTIONS

Consider a series *RC* circuit as shown in Fig. 11.104.

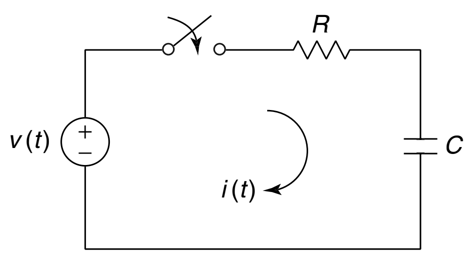



**Fig. 11.104** *RC circuit*

For $t > 0$, the transformed network is shown in Fig. 11.105.
Applying KVL to the mesh,

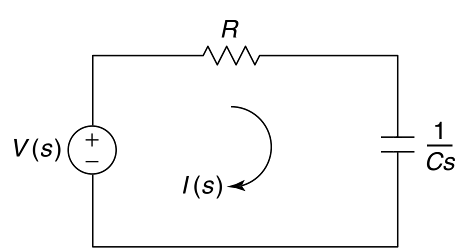



**Fig. 11.105** *Transformed network*

$$V(s) - RI(s) - \frac{1}{Cs}I(s) = 0$$

$$I(s) = \frac{V(s)}{\frac{1}{Cs}+R} = \frac{sV(s)}{R\left(s+\frac{1}{RC}\right)}$$

(a) When unit step signal is applied,

$$v(t) = u(t)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}$$

$$I(s) = \frac{s \times \frac{1}{s}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R\left(s + \frac{1}{RC}\right)}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R} e^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

(b) When unit ramp signal is applied,

$$v(t) = r(t) = t$$

Taking Laplace transform,

$$V(s) = \frac{1}{s^2}$$

$$I(s) = \frac{s \times \frac{1}{s^2}}{R\left(s + \frac{1}{RC}\right)} = \frac{\frac{1}{R}}{s\left(s + \frac{1}{RC}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s + \frac{1}{RC}}$$

$$A = s\, I(s)\big|_{s=0} = \left.\frac{\frac{1}{R}}{s + \frac{1}{RC}}\right|_{s=0} = C$$

$$B = \left(s + \frac{1}{RC}\right) I(s)\Big|_{s=-\frac{1}{RC}} = \left.\frac{\frac{1}{R}}{s}\right|_{s=-\frac{1}{RC}} = -C$$

$$I(s) = \frac{C}{s} - \frac{C}{s + \frac{1}{RC}}$$

Taking inverse Laplace transform,

$$i(t) = C - Ce^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

(c) When unit inpulse signal is applied,

$$v(t) = \delta(t)$$

Taking Laplace transform,

$$V(s) = 1$$

$$I(s) = \frac{s}{R\left(s+\frac{1}{RC}\right)} = \frac{s+\frac{1}{RC}-\frac{1}{RC}}{R\left(s+\frac{1}{RC}\right)} = \frac{1}{R}\left(1-\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}\left[\delta(t) - \frac{1}{RC}e^{-\frac{1}{RC}t}\right] \qquad \text{for } t > 0$$

**Example 11.102** *A rectangular voltage pulse of unit height and T-seconds duration is applied to a series RC network at t = 0. Obtain the expression for the current i(t). Assume the capacitor to be initially uncharged.*

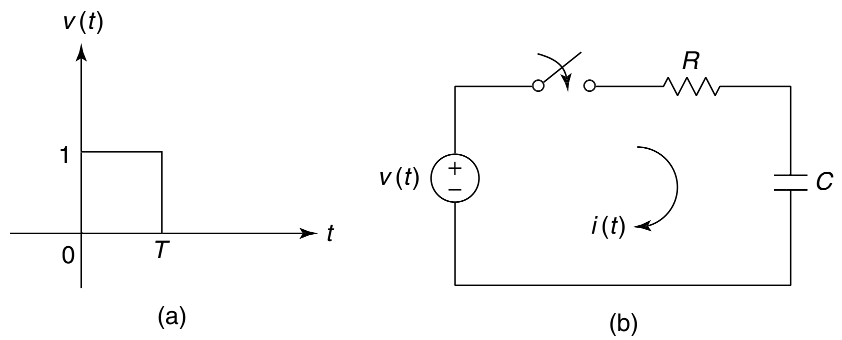



**Fig. 11.106**

**Solution**

$$v(t) = u(t) - u(t-T)$$

$$V(s) = \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1-e^{-sT}}{s}$$

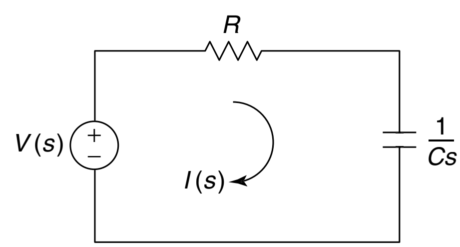



**Fig. 11.107**

For $t > 0$, the transformed network is shown in Fig. 11.107.
Applying KVL to the mesh for $t > 0$,

$$V(s) - RI(s) - \frac{1}{Cs}I(s) = 0$$

$$I(s) = \frac{V(s)}{R+\frac{1}{Cs}} = \frac{\frac{1}{R}s}{s+\frac{1}{RC}}V(s) = \frac{1-e^{-sT}}{R\left(s+\frac{1}{RC}\right)} = \frac{1}{R}\left[\frac{1}{s+\frac{1}{RC}} - \frac{e^{-sT}}{s+\frac{1}{RC}}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}\left[e^{-\left(\frac{1}{RC}\right)t}u(t) - e^{-\left(\frac{1}{RC}\right)(t-T)}u(t-T)\right] \qquad \text{for } t > 0$$

**Example 11.103** *For the network shown in Fig. 11.108, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

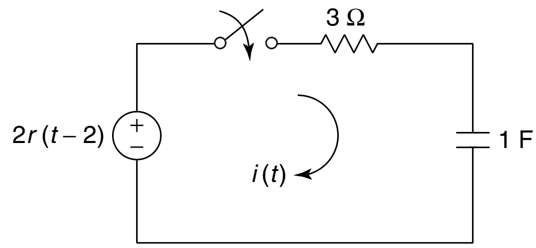



Fig. 11.108

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.109. Applying KVL to the mesh for $t > 0$,

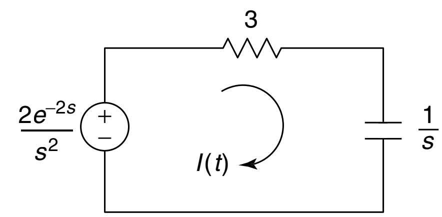



Fig. 11.109

$$\frac{2e^{-2s}}{s^2} - 3\,I(s) - \frac{1}{s}I(s) = 0$$

$$\left(3+\frac{1}{s}\right)I(s) = \frac{2e^{-2s}}{s^2}$$

$$I(s) = \frac{2e^{-2s}}{s^2\left(3+\dfrac{1}{s}\right)} = \frac{0.67e^{-2s}}{s(s+0.33)}$$

By partial-fraction expansion,

$$\frac{0.67}{s(s+0.33)} = \frac{A}{s} + \frac{B}{s+0.33}$$

$$A = \left.\frac{0.67}{s+0.33}\right|_{s=0} = 2$$

$$B = \left.\frac{0.67}{s}\right|_{s=-0.33} = -2$$

$$I(s) = e^{-2s}\left(\frac{2}{s} - \frac{2}{s+0.33}\right) = 2\frac{e^{-2s}}{s} - 2\frac{e^{-2s}}{s+0.33}$$

Taking inverse Laplace transform,

$$i(t) = 2u(t-2) - 2e^{-0.33(t-2)}u(t-2) \qquad \text{for } t > 0$$

**Example 11.104** *For the network shown in Fig. 11.110, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

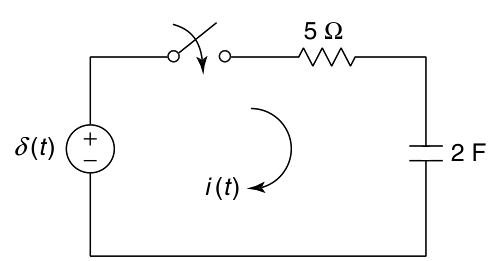



Fig. 11.110

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.111.

Applying KVL to the mesh for $t > 0$,

$$1 - 5\,I(s) - \frac{1}{2s}I(s) = 0$$

$$\left(5 + \frac{1}{2s}\right)I(s) = 1$$

$$I(s) = \frac{1}{5 + \frac{1}{2s}}$$

$$= \frac{2s}{10s+1}$$

$$= \frac{0.2s}{s+0.1}$$

$$= \frac{0.2(s+0.1-0.1)}{s+0.1}$$

$$= 0.2\left(1 - \frac{0.1}{s+0.1}\right)$$

$$= 0.2 - \frac{0.02}{s+0.1}$$

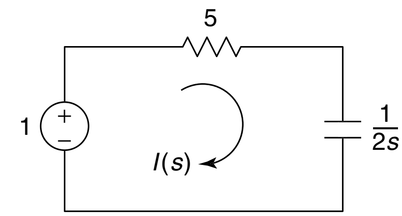



**Fig. 11.111**

Taking inverse Laplace transform,

$$i(t) = 0.2\delta(t) - 0.02\,e^{-0.1t}u(t)$$

**Example 11.105** *For the network shown in Fig. 11.112, find the response $v_0(t)$.*

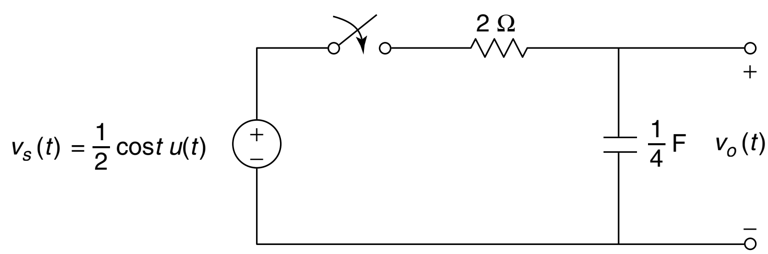



**Fig. 11.112**

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.113.

$$V_s(s) = \frac{1}{2}\frac{s}{s^2+1}$$

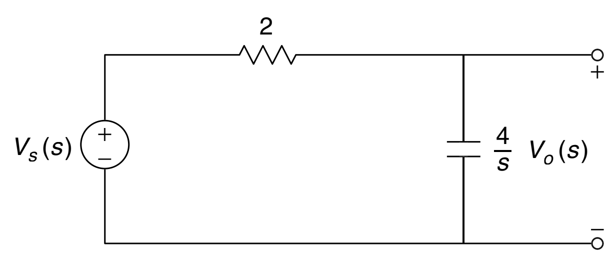



**Fig. 11.113**

By voltage-division rule,

$$V_o(s) = V_s(s) \times \frac{\frac{4}{s}}{2 + \frac{4}{s}} = \frac{2V_s(s)}{s+2} = \frac{s}{(s^2+1)(s+2)}$$

By partial-fraction expansion,

$$V_o(s) = \frac{As+B}{s^2+1} + \frac{C}{s+2}$$

$$s = (As+B)(s+2) + c(s^2+1)$$

$$s = (A+C)s^2 + (2A+B)s + (2B+C)$$

Comparing coefficient of $s^2$, $s$ and $s^0$,

$$A + C = 0$$
$$2A + B = 1$$
$$2B + C = 0$$

Solving the equations,

$$A = 0.4$$
$$B = 0.2$$
$$C = -0.4$$

$$V_o(s) = \frac{0.4s + 0.2}{s^2 + 1} - \frac{0.4}{s+2} = \frac{0.4s}{s^2+1} + \frac{0.2}{s^2+1} - \frac{0.4}{s+2}$$

Taking the inverse Laplace transform,

$$i(t) = 0.4\cos t + 0.2\sin t - 0.4e^{-2t} \qquad \text{for } t > 0$$

**Example 11.106** *Find the impulse response of the voltage across the capacitor in the network shown in Fig. 11.114. Also determine response $v_c(t)$ for step input.*

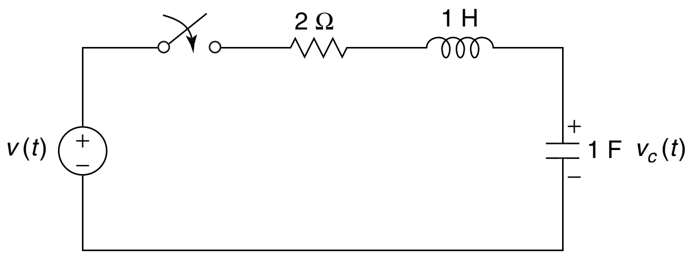



**Fig. 11.114**

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.115.

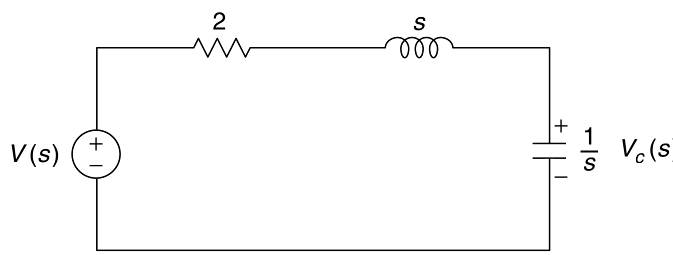



**Fig. 11.115**

By voltage-division rule,

$$V_c(s) = V(s) \times \frac{\frac{1}{s}}{2 + s + \frac{1}{s}}$$

$$= \frac{V(s)}{s^2 + 2s + 1} = \frac{V(s)}{(s+1)^2}$$

(a) For impulse input,

$$V(s) = 1$$

$$V_c(s) = \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$v_c(t) = te^{-t}u(t) \qquad \text{for } t > 0$$

(b) For step input,

$$V(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s(s+1)^2}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$1 = A(s+1)^2 + Bs(s+1) + Cs$$
$$= A(s^2+2s+1) + B(s^2+s) + Cs$$
$$= (A+B)s^2 + (2A+B+C)s + A$$

Comparing coefficient of $s^2$, $s^1$ and $s^0$,

$$A = 1$$
$$A + B = 0$$
$$B = -A = -1$$
$$2A + B + C = 0$$
$$C = -2A - B = -2 + 1 = -1$$
$$V_c(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$v_c(t) = u(t) - e^{-t}u(t) - te^{-t}u(t)$$
$$= (1 - e^{-t} - te^{-t})ut \qquad \text{for } t > 0$$

**Example 11.107** *For the network shown in Fig. 11.116, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

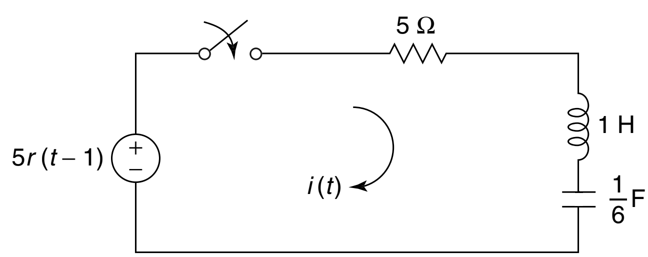



**Fig. 11.116**

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.117.

Applying KVL to the mesh for $t > 0$,

$$\frac{5e^{-s}}{s^2} - 5I(s) - sI(s) - \frac{6}{s}I(s) = 0$$

$$5I(s) + sI(s) + \frac{6}{s}I(s) = \frac{5e^{-s}}{s^2}$$

$$I(s) = \frac{5e^{-s}}{s(s^2+5s+6)} = \frac{5e^{-s}}{s(s+3)(s+2)}$$

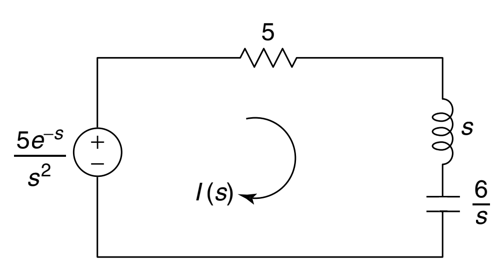



**Fig. 11.117**

By partial-fraction expansion,

$$\frac{1}{s(s+3)(s+2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$A = \left.\frac{1}{(s+3)(s+2)}\right|_{s=0} = \frac{1}{6}$$

$$B = \left.\frac{1}{s(s+2)}\right|_{s=-3} = \frac{1}{3}$$

$$C = \left.\frac{1}{s(s+3)}\right|_{s=-2} = -\frac{1}{2}$$

$$I(s) = 5e^{-s}\left[\frac{1}{6s} + \frac{1}{3(s+3)} - \frac{1}{2(s+2)}\right] = \frac{5}{6}\frac{e^{-s}}{s} + \frac{5}{3}\frac{e^{-s}}{s+3} - \frac{5}{2}\frac{e^{-s}}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = \frac{5}{6}u(t-1) + \frac{5}{3}e^{-3(t-1)}u(t-1) - \frac{5}{2}e^{-2(t-1)}u(t-1) \qquad \text{for } t > 0$$

**Example 11.108** *For the network shown in Fig. 11.118, the switch is closed at t = 0. Determine the current i(t) assuming zero initial conditions.*

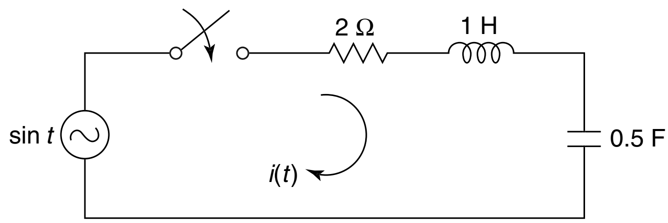



**Fig. 11.118**

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.119.

Applying KVL to the mesh for $t > 0$,

$$\frac{1}{s^2+1} - 2I(s) - s\,I(s) - \frac{2}{s}I(s) = 0$$

$$\left(2 + s + \frac{2}{s}\right)I(s) = \frac{1}{s^2+1}$$

$$I(s) = \frac{s}{(s^2+1)(s^2+2s+2)}$$

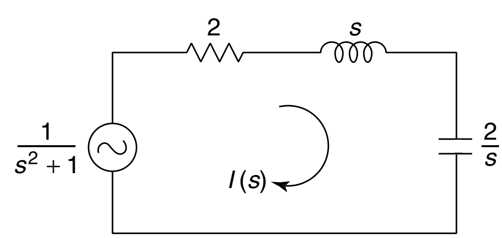



**Fig. 11.119**

By partial-fraction expansion,

$$I(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

$$s = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1)$$

$$= As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + Cs + Ds^2 + D$$

$$= (A+C)s^3 + (2A+B+D)s^2 + (2A+2B+C)s + (2B+D)$$

Comparing coefficients of $s^3$, $s^2$, $s^1$ and $s^0$,

$$A + C = 0$$

$$2A + B + D = 0$$

$$2A + 2B + C = 1$$

$$2B + D = 0$$

Solving these equations,

$$A = 0.2,\ B = 0.4,\ C = -0.2,\ D = -0.8$$

$$I(s) = \frac{0.2s + 0.4}{s^2 + 1} - \frac{0.2s + 0.8}{s^2 + 2s + 2}$$

$$= \frac{0.2s}{s^2 + 1} + \frac{0.4}{s^2 + 1} - \frac{0.2s + 0.2 + 0.6}{(s+1)^2 + (1)^2}$$

$$= \frac{0.2s}{s^2 + 1} + \frac{0.4}{s^2 + 1} - \frac{0.2(s+1)}{(s+1)^2 + 1} - \frac{0.6}{(s+1)^2 + 1}$$

Taking inverse Laplace transform,

$$i(t) = 0.2\cos t + 0.4\sin t - 0.2\,e^{-t}\cos t - 0.6\,e^{-t}\sin t$$

$$= 0.2\cos t + 0.4\sin t - e^{-t}(0.2\cos t + 0.6\sin t) \qquad \text{for } t > 0$$

**Example 11.109** *For the network shown in Fig. 11.120, the switch is closed at t = 0. Determine the current i(t) assuming zero initial conditions in the network elements.*

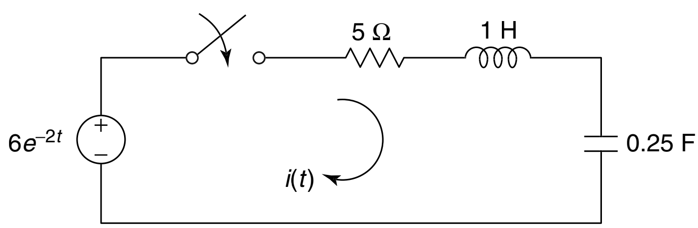



**Fig. 11.120**

**Solution** For $t > 0$, the transformed network is shown in Fig. 11.121.
Applying KVL to the mesh for $t > 0$,

$$\frac{6}{s+2} - 5I(s) - s\,I(s) - \frac{4}{s}I(s) = 0$$

$$\left(5 + s + \frac{4}{s}\right)I(s) = \frac{6}{s+2}$$

$$I(s) = \frac{6s}{(s+2)(s^2 + 5s + 4)}$$

$$= \frac{6s}{(s+2)(s+1)(s+4)}$$

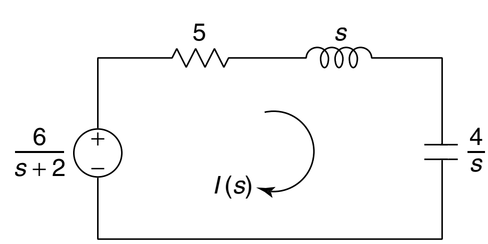



**Fig. 11.121**

By partial-fraction expansion,

$$I(s) = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = (s+2)I(s)\,|_{s=-2} = \left.\frac{6s}{(s+1)(s+4)}\right|_{s=-2} = 6$$

$$B = (s+1)I(s)\,|_{s=-1} = \left.\frac{6s}{(s+2)(s+4)}\right|_{s=-1} = -2$$

$$C = (s+4)I(s)\,|_{s=-4} = \left.\frac{6s}{(s+2)(s+1)}\right|_{s=-4} = -4$$

$$I(s) = \frac{6}{s+2} - \frac{2}{s+1} - \frac{4}{s+4}$$

Taking inverse Laplace transform,

$$i(t) = 6e^{-2t}u(t) - 2e^{-t}u(t) - 4\,e^{-4t}u(t) \qquad \text{for } t > 0$$

**Example 11.110** *The network shown has zero initial conditions. A voltage $v_i(t) = \delta(t)$ applied to two terminal network produces voltage $v_o(t) = [e^{-2t} + e^{-3t}]\,u(t)$. What should be $v_i(t)$ to give $v_o(t) = t\,e^{-2t}\,u(t)$?*

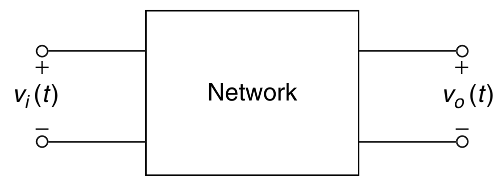



**Fig. 11.122**

**Solution** For $v_i(t) = \delta(t)$,

$$V_i(s) = 1$$

$$v_o(t) = [e^{-2t} + e^{-3t}]u(t)$$

$$V_o(s) = \frac{1}{s+2} + \frac{1}{s+3}$$

System function $H(s) = \dfrac{V_o(s)}{V_i(s)}$

$$= \frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{(s+2)(s+3)} \qquad \ldots\text{(i)}$$

For $v_o(t) = te^{-2t}u(t)$,

$$V_o(s) = \frac{1}{(s+2)^2}$$

From Eq. (i),

$$V_i(s) = \frac{V_o(s)}{H(s)} = \frac{1}{(s+2)^2} \times \frac{(s+2)(s+3)}{2s+5} = \frac{(s+3)}{2(s+2.5)(s+2)}$$

By partial-fraction expansion,

$$V_i(s) = \frac{A}{s+2} + \frac{B}{s+2.5}$$

$$A = 1$$

$$B = -0.5$$

$$V_i(s) = \frac{1}{s+2} - \frac{0.5}{s+2.5}$$

Taking inverse Laplace transform,

$$v_i(t) = e^{-2t} - 0.5e^{-2.5t} \qquad \text{for } t > 0$$

**Example 11.111** *A unit impulse applied to two terminal black box produces a voltage $v_o(t) = 2e^{-t} - e^{-3t}$. Determine the terminal voltage when a current pulse of 1 A height and a duration of 2 seconds is applied at the terminal.*

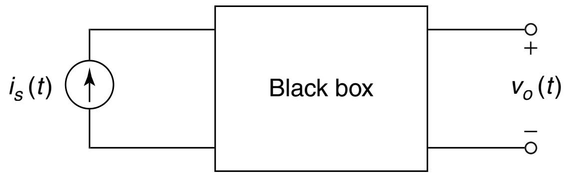



**Fig. 11.123**

**Solution**

$$v_o(t) = 2e^{-t} - e^{-3t}$$

$$V_o(s) = \frac{2}{s+1} - \frac{1}{s+3}$$

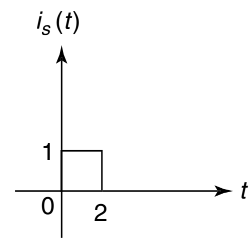



**Fig. 11.124**

When $i_s(t) = \delta(t)$,

$$I_s(s) = 1$$

$$V_o(s) = Z(s)\, I_s(s) \qquad \ldots\text{(i)}$$

$$Z(s) = \frac{V_o(s)}{I_s(s)} = \frac{2}{s+1} - \frac{1}{s+3}$$

When $i_s(t)$ is a pulse of 1 A height and a duration of 2 seconds then,

$$i_s(t) = u(t) - u(t-2)$$

$$I_s(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

From Eq. (i),

$$V_o(s) = \left[\frac{2}{s+1} - \frac{1}{s+3}\right]\left[\frac{1}{s} - \frac{e^{-2s}}{s}\right]$$

$$= \frac{2}{s(s+1)} - \frac{1}{s(s+3)} - \frac{2e^{-2s}}{s(s+1)} + \frac{e^{-2s}}{s(s+3)}$$

$$= 2\left[\frac{1}{s} - \frac{1}{s+1}\right] - \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right] - 2e^{-2s}\left[\frac{1}{s} - \frac{1}{s+1}\right] + \frac{e^{-2s}}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

Taking the inverse Laplace transform,

$$v(t) = 2[u(t) - e^{-t}u(t)] - \frac{1}{3}[u(t) - e^{-3t}u(t)] - 2[u(t-2) - e^{-(t-2)}u(t-2)] + \frac{1}{3}[u(t-2) - e^{-3(t-2)}u(t-2)]$$

$$\text{for } t > 0$$

# Exercises

**11.1** Find $L\{f'(t)\}$ of $f(t) = \left(\dfrac{1-\cos 2t}{t}\right)$

$$\left[s\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$$

**11.2** Find Laplace transform of the follwoing function:

$f(t) = t + 1 \quad 0 \le t \le 2$

$\quad = 3 \qquad t > 2$

$$\left[\frac{1}{s}(1-e^{-2s})\right]$$

**11.3** For the network shown in Fig 11.125, the switch is closed at $t = 0$. Find the current i1($t$) for $t > 0$.

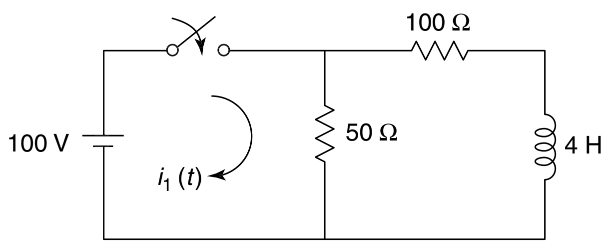



**Fig. 11.125**

$[i_1(t) = 3 - e^{-25\,t}]$

**11.4** Determine the current $i(t)$ in the network of Fig. 11.126, when the switch is closed at $t = 0$. The inductor is initially unenergized.

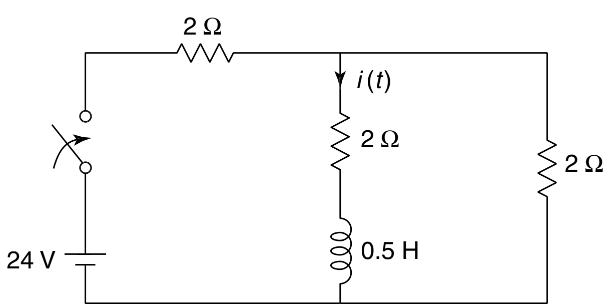



**Fig. 11.126**

$[i(t) = 4(1 - e^{-6t})]$

**11.5** In the network of Fig. 11.127, after the switch has been in the open position for a long time, it is closed at $t = 0$. Find the voltage across the capacitor.

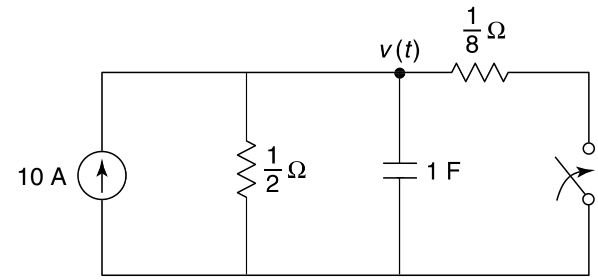



**Fig. 11.127**

$[v(t) = 1 + 4\,e^{-10t}]$

**11.6** The circuit of Fig. 11.128, has been in the condition shown for a long time. At $t = 0$, switch is closed. Find $v(t)$ for $t > 0$.

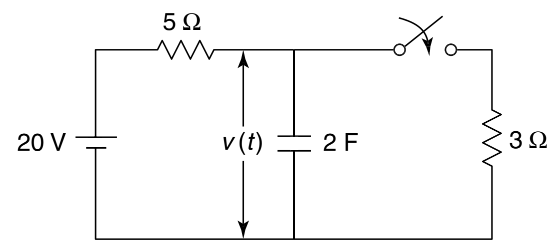



**Fig. 11.128**

$[v(t) = 7.5 + 12.5\,e^{-(4/15)t}]$

**11.7** Figure 11.129 shows a circuit which is in the steady-state with the switch open. At $t = 0$, the switch is closed. Determine the current $i$ ($t$). Find its value at $t = 0.114$ μ seconds.

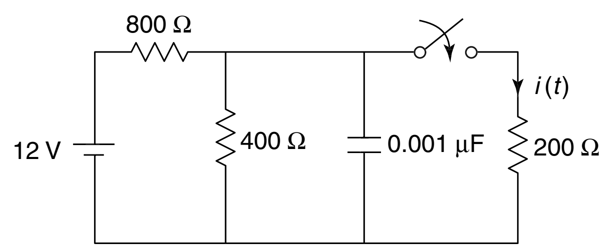



**Fig. 11.129**

$[i(t) = 0.00857 + 0.01143\,e^{-8.75\times 10^6 t}, 0.013\text{ A}]$

**11.8** Find $i(t)$ for the network shown in Fig. 11.130.

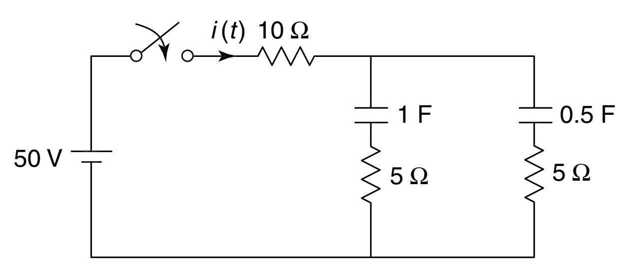



**Fig. 11.130**

$[i(t) = 0.125\ e^{-0.308t} + 3.875\ e^{-0.052t}]$

**11.9** Determine $v(t)$ in the network of Fig. 11.131 where $i_L(0^-) = 15$ A and $v_c(0^-) = 5$ V.

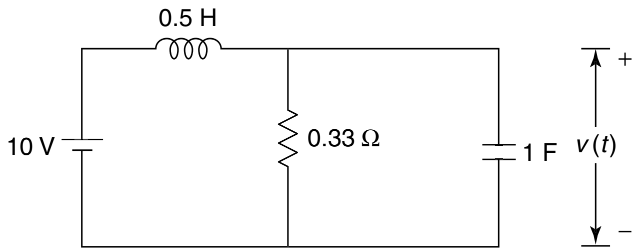



**Fig. 11.131**

$[v(t) = 10 - 10e^{-t} + 5e^{-2t}]$

**11.10** The network shown in Fig. 11.132 has acquired steady state with the switch at position 1 for $t < 0$. At $t = 0$, the switch is thrown to the position 2. Find $v(t)$ for $t > 0$.

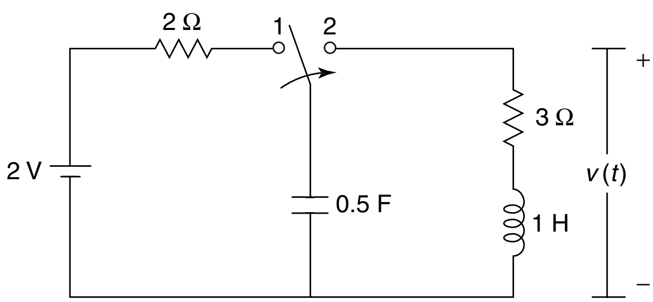



**Fig. 11.132**

$[v(t) = 4e^{-t} - 2\ e^{-2t}]$

**11.11** In the network shown in Fig. 11.133, the switch is closed at $t = 0$. Find current $i_1(t)$ for $t > 0$.

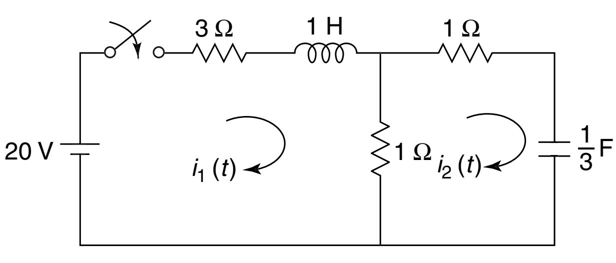



**Fig. 11.133**

$[i_1(t) = 5 + 5e^{-2t} - 10e^{-3t}]$

**11.12** In the network shown in Fig. 11.134, the switch is closed at $t = 0$. Find the current through the 30 Ω resistor.

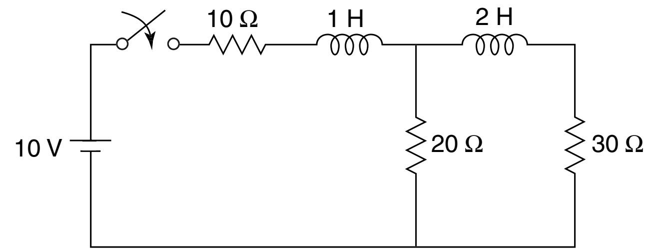



**Fig. 11.134**

$[i(t) = 0.1818 - 0.265\ e^{-13.14t} + 0.083\ e^{-41.86t}]$

**11.13** The network shown in Fig. 11.135 is in steady state with $s_1$ closed and $s_2$ open. At $t = 0$, $s_1$ is opened and $s_2$ is closed. Find the current through the capacitor.

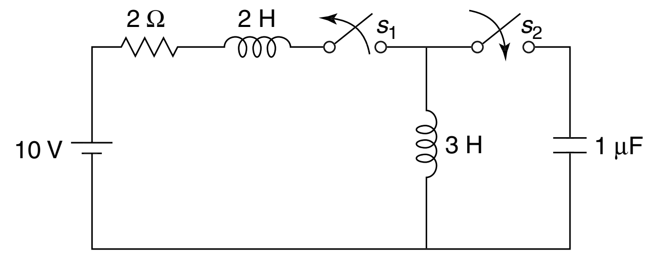



**Fig. 11.135**

$[i(t) = 5 \cos (0.577 \times 10^3\ t)]$

**11.14** In the network shown in Fig. 11.136, find currents $i_1(t)$ and $i_2(t)$ for $t > 0$.

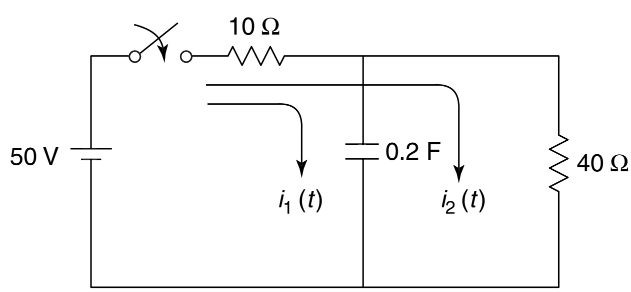



**Fig. 11.136**

$[i_1(t) = 5\ e^{-0.625t},\ i_2(t) = 1 - e^{-0.625t}]$

**11.15** For the network shown in Fig. 11.137, find currents $i_1(t)$ and $i_2(t)$ for $t > 0$.

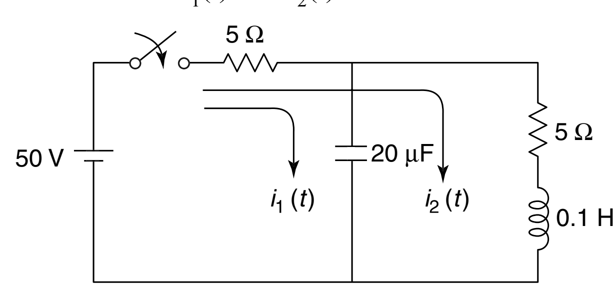



**Fig. 11.137**